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SOME PROPOSITIONS

IN

GEOMETRY.

BY JOHN HARRIS.

IN FIVE PARTS.

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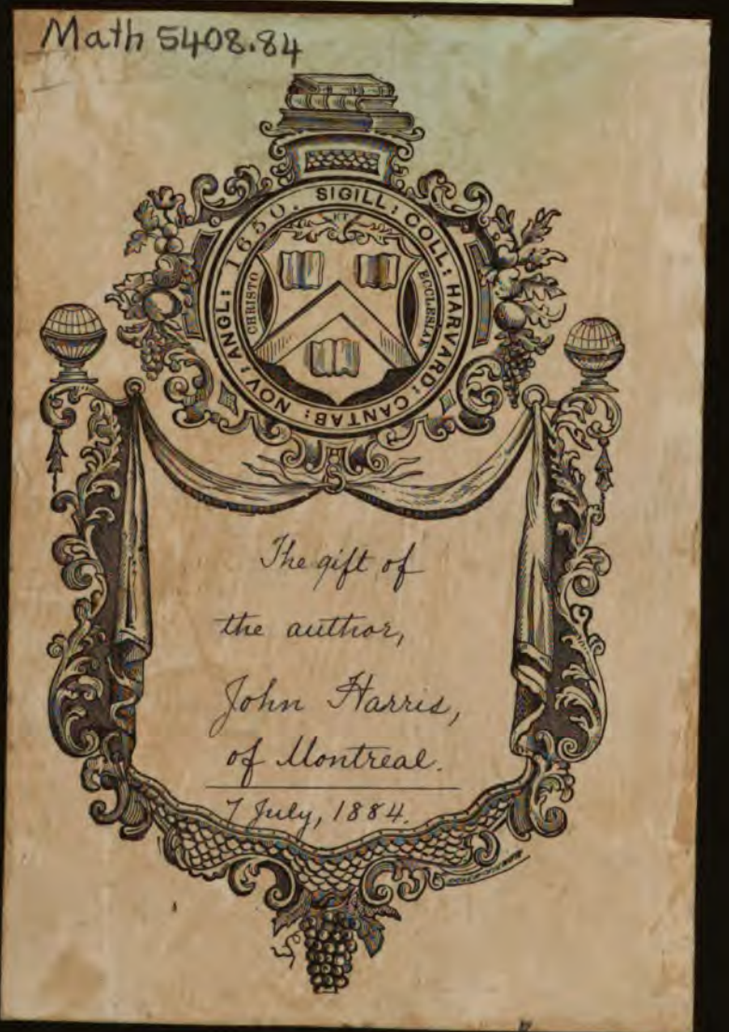

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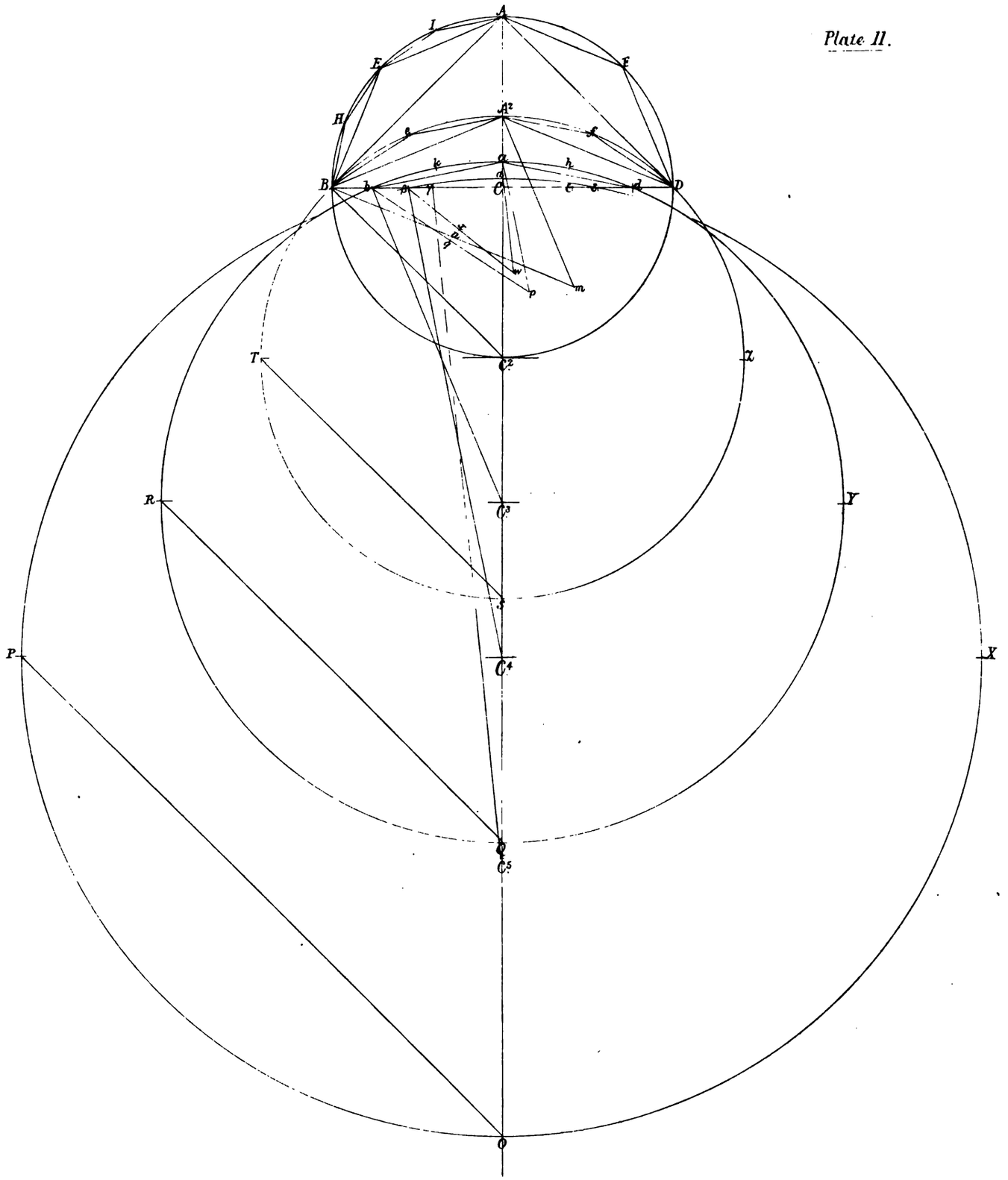

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IN

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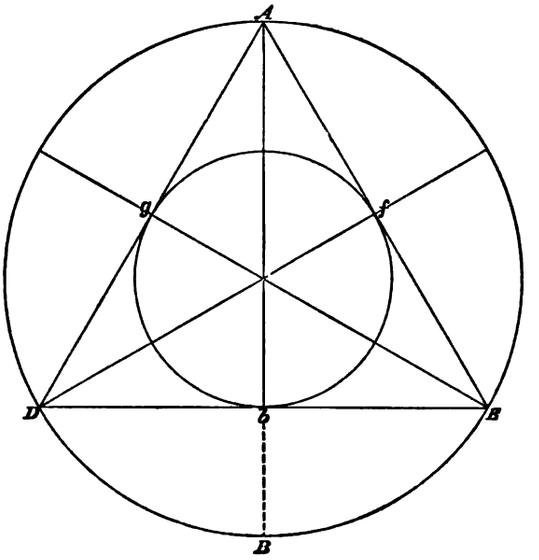

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LONDON:
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TO THE

GREAT AUTHOR OF THE CIRCLE

THIS ENDEAVOUR

TO ADD SOMETHING OF VALUE

TO THE SCIENCE OF FORM AND MAGNITUDE

AS BEQUEATHED TO US

BY THE PHILOSOPHERS OF THE ANCIENT CIVILIZATION

IS MOST REVERENTLY

INSCRIBED.

PREFACE.

THOSE who have read our previous observations* on the true character of "Geometry," as set forth, applied, and illustrated, in the work called "Euclid's Elements of Geometry," will apprehend at once the purpose of the earlier and more important of the definitions which precede the propositions contained in the present volume.

For the benefit of readers who have not read, or may not remember, those observations, we will here very briefly re-state the argument propounded in them; namely, that "Euclid's Elements" constitutes an arranged system of inductive reasoning, applied to, and illustrated by its application to, the subjects of the departmental science of *Form and Magnitude.*

The great characteristic of the system (and in this becomes apparent its harmony and close alliance with the modern inductive system known as the Baconian) is, that it is strictly based upon *fact.* The fundamental doctrine, primarily inculcated as imperative, and prominently exemplified throughout Euclid's work, is that *certainty* can be obtained as the result of reasoning only when that reasoning starts from and rests upon the *truth* of *fact and reality* as its basis.

Now, assuming that this view of the nature of "Geometry" will bear careful consideration, will commend itself to the candid and intelligent student as most reasonable, and eventually meet with general acceptance (and we do not hesitate to assume that it will do so), Geometry may

* At the commencement of the Treatise on the "Circle and Straight Line," and in the "Introductory Remarks to Lecture First."

evidently be considered as the converse of and opposed to that system of reasoning known to most educated persons as "Metaphysics": for "Metaphysics" is a system based, whenever it appears to have an actual basis, on *supposition*. Not, however upon demonstrable suppositions or *theorems*, which, when geometrically demonstrated, are equivalent to *ideal facts*, but upon undemonstrable theorems at best, and usually upon vague suppositions of an unreal character, fallacious assumptions, and inferences of an arbitrary, and more or less irrational, kind.

But although this (opposition in the distinctive characteristics of the respective bases) constitutes the great fundamental difference between the two systems—"Geometry and Metaphysics," it does not nearly comprise the whole difference; for, in its mode of procedure, Geometry loves *Law* and *Order*, and Metaphysics detests both. Geometry delights in clear daylight, and desires, right or wrong, to speak *definitely* with the utmost distinctness, and make itself understood; whereas Metaphysics seeks the dark cloud and the dense mist in which to ensconce itself, speaks *indefinitely* in a language of its own, in which each word has almost any signification it may at the moment suit the purpose of the speaker to bestow upon it, and prides itself especially on utterly confounding those who most attentively listen to its arguments.

Wherefore, then, since *fact, reality, and truth* are the foundations of sound science, and *just reasoning* its great weapon of attack and defence, "Geometry" must constitute the bulwark of Sound-Science against open assaults upon its domain by Unsound-Science, and also against the secret endeavours of that persistent and ever-watchful foe to enter its territory and pervert the unwary amongst its adherents.

Taking this wide but definite signification of the distinctive expression, "Geometry," we are met by the question: Are "Euclid's Elements" and

"Geometry" synonymous or equivalent terms? To this the reply is—No. A good deal of "Geometry" is comprised in "Euclid's Elements"; also the latter may be justly considered as a very able exposition of the principles of sound inductive reasoning; and very much of it certainly belongs to Geometry in the wide and strict sense. *Not all of it?* No; a part of it, so far from belonging to Geometry, belongs to the great antagonist of Geometry—Metaphysics. And this part, albeit a very small part, occupies the vantage ground of a first speaker: it stands in the forefront, and first addresses the student. The work literally commences with it; for it is that part known as the three primary definitions —defining *the point* and *the line.*

What has been the effect of this flaw, so small in bulk or quantity of matter, occurring at the very commencement of Euclid's great work? The effect has been, as we have already pointed out (in the former notices referred to), to give a dubious character to the whole work; to bring a metaphysical cloud over Geometry's choicest exponent, and to cause the expression "Geometry" itself to denote a mystical and unreal department of knowledge . . a shadowy region, outside the confines of natural science and far away from the domain of reality.

The purpose, then, of a part of the definitions prefixed to the contents of the following pages, is the correction of that which we look upon as a serious and most important error comprised in the three primary definitions of "Euclid's Elements." * It is by the substitution of primary definitions, consistent and harmonious with the strict sound system of inductive reasoning set forth in the "Elements," for those which (according to our views) are quite inconsistent and discordant with that system, that what

* But not wholly confined to the first three definitions; as will appear on consideration of those which we propose to substitute for them.

is now amiss in that important work can be set right; and, accordingly, we propose the definitions which follow, and submit them to the judgment of the mathematical reader, in the belief that they will be found suitable and well adapted for the purpose.

That purpose applies, however, to a part only of our definitions. Of the remainder, some are intended to supply supposed wants in nomenclature, and others are merely suggestions for rendering more definite and concise a few of the elementary propositions of Geometry. To prevent renewed misunderstanding, it may be well for us to state distinctly that none of these definitions are intended to instruct experienced and highly educated mathematicians in the A B C of Elementary Geometry; they are submitted simply and solely for the express purposes we have just endeavoured to explain.

Moreover, with two (or more) comparatively trifling exceptions,* they (the definitions) are quite independent of the Propositions which follow them. Those Propositions, as will readily appear on examination, do not require any additions to or departures from the well known Definitions, Postulates, and Axioms contained in Euclid's work. And, in saying that our Definitions are quite independent of the Propositions which follow them, the meaning is included that the latter are likewise quite independent of the former.

London, Kilburn Square,
13th September, 1881.†

* The *First* and *Second* Propositions are almost necessitated by the definitions to which they have reference, and they expressly belong to those specified definitions.

† In consequence of important additions, in the first instance, and subsequently, of a prolonged interruption, publication has been most unwillingly delayed from the above date until the present, viz., April, 1884.

INTRODUCTORY REMARKS.

HAS mathematics become a wholly practical business: so that theoretical speculation on mathematical subjects is now not only an antiquated but also an unnecessary and useless employment? We have been assured so, directly and indirectly, by more than one or two eminent mathematicians. "Modern methods get easily and readily at the result; why, then, trouble ourselves with the tiresome speculations of ancient philosophy?"

Here we are reminded of an eminent practical astronomer whose attention we innocently invited to an argument as to the position (parallelism or obliquity) of the earth's axis of rotation, relatively to that of the sun. We were immediately assured by him, not only seriously and with evident sincerity, but also with something of the vehemency of indignation, that his time was now, and the greater part of his life had been, employed upon the important practical work of astronomy, and it was not likely he could give his attention for a single moment to such elementary *theories*. Observe that he did not say *trifles*, nor did he say *nonsense*, but, perhaps from a charitable disinclination to hurt our feelings, he said *elementary theories*.

So in a letter (a very kindly and courteous letter) from an eminent mathematician, of which we are here also mindful, the speculations of the ancient geometers were spoken of quite placidly and approvingly, not as follies discreditable to men of science at that early period, but as belonging to the infancy or childhood of science when they were *in place*, useful, and, perhaps, even, necessary.*

It is not indeed easy to read off the idiosyncrasies of some mathematicians on a

* The letter of our esteemed correspondent had especial reference to the ratio of the circle's circumference to the diameter, and the speculations on that subject of one who ranked high amongst the most renowned philosophers of the ancient school.

Now the ancient speculations on that subject, as in many other cases, had a practical outcome and result. Modern methods have also applied themselves to the same subject and got at a result,

merely superficial inspection; for "the Mathematician" is, *sui generis*, (as a florist would express it) of sorts.

There is . . the very positive and readily incensed Mathematician. He is very positive, for instance, "that a line has no extremities because a point has no existence," and would send to the stake or the gibbet, without even the formality of trial, any vile heretic who dared to breathe a doubt as to so vital and fundamental a dogma.

There is . . the highly exalted Mathematician: so high up that he scarcely deigns to consider whether there be anything so commonplace as the *terra firma* of reality beneath his feet.

There is . . the profound Mathematician: to whom ideas expressed in anything but algebraic formulæ and the choice phraseology of technical mathematics, are as worthless chaff and "infinitely contemptible."

There is . . the exclusive orthodox Mathematician: with whom "Mathematics" is a sort of privileged religion, having its special *articles* and technical dogmas. None but the initiated must venture to enter its temple, and woe be to him who dares to do so without a formal certificate from its priests. One of his most valued dogmas is that $\pi = 3{\cdot}14159\ldots$ This is called an approximation; and which if a man do not faithfully believe, he will inevitably go . . Ah, well! never mind where he will go to, but, at all events, no Mathematician must hold converse or communication with such a profane person.

---

upon the *assumed* correctness of which, although they have called it an approximation, they have been wont to plume themselves exceedingly as a supreme triumph and proof of superiority.

Their *Approximative* result is that . . . . $\pi = 3{\cdot}14159$ ..

The result of the ancient *Speculator* was that $\pi = \frac{22}{7} = 3{\cdot}142857\ldots$

The actual ratio having now been recently obtained by geometry and indisputably established by demonstration; we have $\pi = 3{\cdot}142696\ldots$

Because it is shown that, when it expresses the true ratio, $\pi = \sqrt{8} \times \frac{10}{9} = 3{\cdot}142696\ldots \left(\text{or } \sqrt{50} \times \frac{4}{9}\right)$

When it is fully apprehended that this is *indisputably* the true state of the case, it will then appear that the ancient result was *very nearly* correct; much more nearly than the modern: and hence that, with regard to this very important practical result at least, modern methods must still bow the head to the great speculative outcomes of ancient science.

And there is the more liberal, genial, and sensible Mathematician . . But, after all: What is a *Mathematician?*

For example, A man who knows how to apply to the practical purposes of computation and analysis, and is able to use with facility, ingeniously contrived mathematical instruments (Methods), is, unquestionably, able to render himself a very useful man. He possesses valuable practical knowledge, and by the application of it may become very potential in aiding the work of science, and so make his life highly serviceable to his fellow-men. But is *he* a *Mathematician?* The question may perhaps be best answered by another question. Has he himself constructed some one or more of those valuable mathematical instruments? Has he himself invented any of the powerful labour-saving contrivances in the practical application of which he is an adept? If he *has*, then we have no hesitation in affirming that he *is* a *Mathematician.*

We have once again dared a charge of presumption, by venturing to propose and publish the definitions which immediately follow these remarks; and now, referring the reader to our observations on the true nature of Geometry, already published, we will end this Introduction, in a climax of temerity, by defining a *Mathematician.* Thus:—A Mathematician is one in whom is combined justness of reasoning and originality of thought. For instance, *Sir Humphrey Davy.*

The value of this particular example as an illustration, however, is much dependent on the correctness of our estimate as to the place Davy's name will eventually occupy, amongst those of England's most illustrious worthies, on the scroll of fame.

According to our individual judgment, amongst those Englishmen who, in the recent period, have devoted their lives and abilities to the service of physical science, Sir Humphrey Davy stands intellectually a head and shoulders higher than any one of his contemporaries or successors. Now, then, if such estimate be substantially correct, we would, as an equivalent mode of stating the circumstance, say that Humphrey Davy is the greatest Mathematician of those Englishmen who in the recent period have devoted themselves to the service of Natural Science. *What! Humphrey Davy the great chemist?* No. But Humphrey Davy the great *philosopher*, and *therefore* Humphrey Davy the great *Mathematician.*

THE SCIENCE OF FORM AND MAGNITUDE

AS A DEPARTMENT

OF IDEAL AND OF NATURAL SCIENCE.

THE SCIENCE OF FORM AND MAGNITUDE.

THE student commencing the study of Geometry is sometimes taught to regard (or that he may regard) the surfaces, lines, and points of geometry as the mental representations of those substantial or *naturally real* surfaces, lines, and points with which experience has already made him familiar. This, however, may lead to fundamental misapprehension of a very grave character; for, indeed, it reverses the true and actual relationship of Ideal to Natural Science. Certainly, the *natural* reality may be used or referred to as an illustration of the *ideal* reality—thus the approximately even surface of a table may serve to illustrate a geometrical plane superficies; and actual marks and lines on a natural surface, such as a sheet of paper or the face of a board, are almost indispensably necessary to *naturally* specify, determine, and communicate the *ideal actualities* which they represent and illustrate. But let it be distinctly understood and carefully remembered that the higher and more nearly absolute reality is the *ideal*, which has been adapted, by means of what in the higher sense are *artificial* limitations, to those conditions which seem to us *natural*, by the Creator, to serve the special purposes of human terrestrial existence; and, hence, these adaptations and limitations in the aggregate may be considered to constitute that which we distinctively call "Nature." The Science of Form and Magnitude therefore, although belonging also to Natural Science as a Department thereof, belongs to Ideal Science much more directly than other departmental natural sciences, such as Astronomy, Chemistry, Physiology, etc.

DEFINITE NOMENCLATURE—(OF THE DEPARTMENTAL SCIENCE OF "FORM AND MAGNITUDE").

OF POINTS.

(Def. 1.) A *Point* . . is a locality in space, which, being selected and made specific by the imagination, becomes ideally existent, *as a point*, but does not include extension (*i.e.*, has no dimension). A moving or movable point is an existent point which changes its locality under the laws of geometry.

OF LINES.

(2.) *Longitudinal Extension* . . is the space which directly intervenes between two separated points. A *Line* . . is longitudinal extension which, being made specific by the imagination, becomes ideally existent.

(3.) A *Finite Line* . . is that in which the distance from each other of the two extreme points, connected by the line, is limited. This is expressed by saying: a *finite* line is a line of which the *magnitude* is limited.

A *Definite Line* . . is that of which the magnitude (or distance between the separated points) is definitely limited. An *indefinite* line is that of which the magnitude, although limited, is not definitely limited. An *Extended Line* . . is a line extended beyond one or both of the extreme points of the finite line. A line so extended is called a *produced* line; and, when extended through one, or both, of the extreme points to a point, or to points, each at a finite distance from the primary extreme point, the line becomes a finite produced line.

An *Infinite Line* . . is a produced line of which the extension is endless. That is, when the line is said to be produced *infinitely* in one direction only, the infinite line has no *terminal* point; and, when produced infinitely in both directions, the infinite line has neither *original* nor *terminal* point.

Lines are of two kinds; namely, straight lines and curved lines.

(4.) A *Straight Line** . . is a line which extends with direct evenness; *i.e.*, without curvature or irregularity.

(5.) A simple *Curved Line* . . is a part of the circumference of a circle. The degree of curvature is dependent upon the distance from it of the centre of the circle to which it belongs; which centre is called the *centre* of *description* of the curve, and from that centre all points in the curve are equidistant. All simple curves are therefore called *Cyclal curves.*

A simple or *Cyclal* curve is generated by a movable point, moving equidistantly and evenly in the same plane, around a fixed point. The cyclal curve is then the path of the moving point; the degree of curvature, for a certain quantity of motion, depending upon the *radial* distance of the moving from the fixed point. The path of the moving point eventually returns into itself, thereby enclosing space and forming a complete figure called a *Circle.* Simple curvature may therefore be also called *statico-cyclal* curvature, because the *central* point does not move. All compound curves (*i.e.*, all curves except the simple cyclal) belong to one of two kinds, *dynamico-cyclal* or *radius-varying-cyclal* curvature, or to a compound of these two. In each and every case (including simple curvature) the curve is generated by the motion of one point relatively to a second point in such wise that the motion of the first *tends* to carry it around the second, and does or would do so completely if the second point remain motionless and the *radial* distance unchanged. In dynamico-cyclal curvature the second point also moves,

* It may be suggested that for ordinary purposes the best definition of a *straight line* is . . a line which is *straight.* And similarly of a *curved line* . . a line which is *curved.* An *uneven line* is a line which is not *even;* and an *irregular line* is a line which is neither uniformly straight nor uniformly curved.

and does so with a relative velocity which may be less than, equal to, or greater than that of the first (respectively modifying the curvature). In radius-varying curvature the radial distance of the first from the second (*central*) point may increase or decrease, and, in either case, may do so uniformly or irregularly.

(6.) An *Ultimate Line* . . is one of the ultimate parts resulting from the continued division of a line.*

(7.) A *Secting Line* . . is that which divides the areal magnitude contained in a geometrical figure into parts or sections. The *secting line* converts that part of the areal magnitude through which it passes into specific longitude which constitutes the common lineal boundary of the two sections when considered as separate magnitudes.

(8.) The figures formed by straight lines are called *rectilineal* figures. The figures formed by curved lines are called *curvilineal* figures.

OF AREAL EXTENSION.

(9.) A *Plane*. . When two straight lines cross each other at right angles, the longitudinal and latitudinal *extension*, without altitude and of perfect evenness, which includes the whole and every part of the two straight lines is called a *plane*. A *plane superficies* is a finite plane made specifically existent as a magnitude by the imagination, of which the boundary in each direction is a line.

(10.) Plane extension definitely limited by a lineal boundary is called a *plane areal figure*. The magnitudinal extension is called its *area*, and the lineal boundary its *perimeter*.

OF SOLIDS.

(11.) Magnitudes composed of longitudinal and latitudinal (*i.e., of areal*) combined with altitudinal extension, bounded and limited by superficies, are called *solids*. The entire superficies bounding a solid magnitude, when considered as a whole, but independently of the magnitude bounded by it, is called a *solid superficies*.

(12.) *Primary Solids* are called cubical, spherical, conical, or cylindrical magnitudes.

Secondary Solids are modifications of primary solids, or derivatives from them by section.

Solid figures of one kind are generated by the motion of a plane areal figure which moves from a certain definite plane to another definite plane parallel to the first, and

* An *ultimate* line is separable, only by destructive decomposition, into its ideal constituents,—namely, into *Extension* and *Locality*. This may be considered a theorem of which the demonstration is inherent; for the conception of a *line* includes *extension*, and therefore a line must possess lineal magnitude. Hence it follows;—there must be an existent ultimate line (*lineal magnitude*) which cannot be made less. In other words: Lineal magnitude is *quantitative* and therefore constituted of ultimate parts.

separated from it by intervening space. The space through which the lineal boundary of the superficies passes in moving from the first plane to the second is *ideally* converted into superficial magnitude, and constitutes the boundary of the altitude or thickness of the solid. Examples of this kind are the *Cube* and other *Parallelopipeds*. Solid figures of another kind are generated by the revolution of a plane areal figure around a diameter or a lineal axis which bisects the superficies. The space through which the lineal boundary of the plane superficies passes in its revolution is *ideally* converted into superficial magnitude, and constitutes the boundary of the solid. Examples of this kind are the *Sphere*—generated by the revolution of a circle around its diameter as a lineal axis; the *Cone*—generated by the revolution of an equilateral triangle; the *Cylinder*—generated by the revolution of a right-angled parallelogram.

OF CIRCLES.

(13.) A *Lineal Circle* is a continuous curvilineal boundary enclosing space; of curvature such that the boundary surrounds a certain point equidistant from every part of, and component point in, that boundary. The point so surrounded and equidistant from every part of the curvilineal boundary is called the *centre* of the circle: and the curvature of the boundary containing the circle is called *Cyclal curvature.*

(14.) An *Areal Circle* is the magnitudinal area enclosed and bounded by the lineal circle which surrounds it. The lineal boundary is called the *circumference* or perimeter of the *areal* circle.

(15.) An *Arc of a Circle* is a part of the circle's circumference. *Similar Arcs* are equal fractions of the lineal circles to which the arcs respectively belong.

(16.) The radial line or *radius* with which the description of the arc commences is called the *original* radius of the arc; and the line or radius with which the arc terminates is called the *terminal* radius of the arc. Also the point at the commencement of the arc (*i.e.*, at the extremity of the *original* radius) is called the *original* extremity of the arc: and the point at the opposite extremity of the arc is called the *terminal* extremity of the arc.

OF ANGLES AND TRIANGLES.

(17.) Equality of Angles.—When, of two angles, the arms of the one, have the same inclination one to the other, or one of them to a line perpendicular to the other, which the arms of the other angle have one to the other, or which one of them has to a line perpendicular to the other, the angles are said to be equal: and, also, if the two arms containing each angle are perpendicular one to the other, *i.e.*, are *at right angles*, the angles are equal.

(18.) *Similar angles* are equal angles in which the arms of the one angle have the same ratio one to the other, which the arms of the other angle have correspondingly one to the other.

(19.) *Similar triangles* are those in which the angles in the one are equal respectively to the corresponding angles in the other. (The corresponding angles in the similar triangles, and the corresponding sides in the similar angles, are said to be *respondent.* That is, each is *respondent* to the other with which it corresponds.)*

Hence :—*Equal Angles* are angles of which the arms have equal inclinations.

Similar Angles are equal angles of which the arms have equal magnitudinal ratios.

Similar Angles laterally equal are similar angles of which the *respondent* arms are equal magnitudes.

Similar Triangles are those in which the respondent angles are similar.

Triangles Similar and Equal are those in which the respondent arms of the respondent similar angles are equal magnitudes.

(20.) A point within an equilateral triangle, equidistant from the vertices of all the three angles, is called the *centre* (or *vertex-centre*) of the equilateral triangle. Similarly a point within an isosceles triangle, equidistant from the vertices of all the three angles, is called its *vertex-centre.*

(21.) The point within an isosceles triangle, in which the two lines, drawn from the points of bisection of the two equal sides of the triangle to the vertices of the angles at the base, intersect each other, is called the *lateral-centre* of the triangle.

* The suggestion here intended is to confine the Euclidian expression "homologous" to the similar superficies of solid figures, and to substitute the somewhat more convenient expression "respondent" for the similar sides of plane lineal figures.

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PART FIRST.

ELEMENTARY PROPOSITIONS.

PART FIRST.

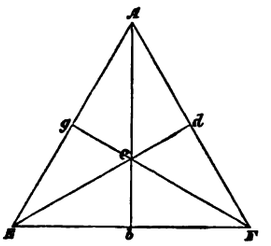
ELEMENTARY PROPOSITIONS.

PROP. I. PROBLEM.

To find the centre (or vertex-centre) of an equilateral triangle. (Def. 19.)

Let $A\,E\,F$ be the given equilateral triangle: it is required to find the centre thereof.

Bisect the angle $E\,A\,F$ in the line $A\,b$ (Euclid, I. 9), the angle $A\,E\,F$ in the line $E\,d$, and the angle $A\,F\,E$ in the line $F\,g$. The point C, in which the three lines intersect, is the centre of the triangle $A\,E\,F$.



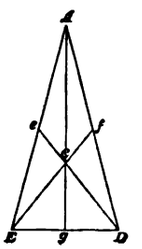
Because the line $A\,b$ is perpendicular to the side $E\,F$, the line $E\,d$ perpendicular to the side $A\,F$, and the line $F\,g$ perpendicular to the side $A\,E$, and because the triangle $A\,E\,F$ is equilateral, the three angles $E\,C\,F$, $A\,C\,F$, $A\,C\,E$, whose vertices meet in the common point C, are similar and equal each to each, therefore the lines $b\,C$, $d\,C$, $g\,C$, which respectively bisect those triangles, are equal each to each. But the whole lines, $A\,b$, $E\,d$, $F\,g$, are equal each to each. Therefore their remainders, $A\,C$, $E\,C$, $F\,C$ are equal each to each, and the point C is shown to be equidistant from the respective vertices A, E, F, of the three angles of the given equilateral triangle. Wherefore the point C is the centre thereof.

PROP. II. PROBLEM.

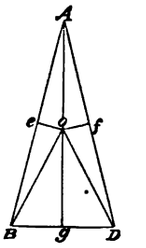
To find the lateral and vertex-centres of an isosceles triangle. (Def. 21.)

Let $A\,E\,D$ be the given isosceles triangle.

Bisect the vertical angle $E\,A\,D$ in the line $A\,g$, and bisect the two sides $A\,E$ and $A\,D$ in the points e and f respectively. Join $E\,f$ and $D\,e$. The point C, in which the three lines $A\,g$, $B\,f$, $D\,e$, intersect, is equidistant from the angle points E and D, and is therefore the lateral centre of the triangle $A\,E\,D$.



To find the vertex-centre of an isosceles triangle. (Def. 21.)

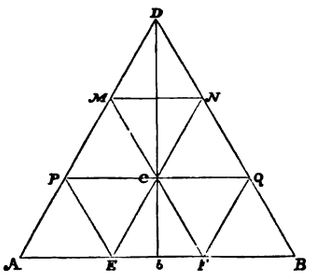


Let $A\,B\,D$ be the isosceles triangle; bisect the vertical angle $B\,A\,D$ in the line $A\,g$, and bisect the two equal sides $A\,B$ and $A\,D$ in the points e and f. From points e and f respectively, at right angles to the sides $A\,B$ and $A\,D$ draw $e\,O$ and $f\,O$ meeting each other and also the line $A\,g$ in the point O. The point O, equidistant from the angle points A, B, D, is therefore the vertex-centre of the isosceles triangle $A\,B\,D$.

Prop. III. Problem.

To divide a given straight line into three equal parts.

Let $A\,B$ be the given straight line. Upon $A\,B$ describe the equilateral triangle $D\,A\,B$ (Euclid I. 1). Find the centre C of the equilateral triangle (Prop. 1.). Through C draw the line $N\,E$ parallel to the side $D\,A$, and the line $M\,F$ parallel to the side $D\,B$ of the triangle, respectively intercepting the line $A\,B$ in the points E and F. The straight line $A\,B$ shall be divided into three equal parts in the points E and F.



From the point E, parallel to $D\,B$, draw $E\,P$ intercepting $D\,A$ in P; and from the point F, parallel to $D\,A$, draw $F\,Q$ intercepting $D\,B$ in Q. Join $M\,N$; and through C, join $P\,Q$.

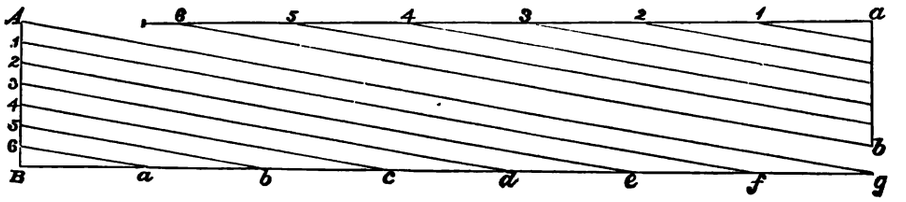
Because $M\,F$ is parallel to $D\,B$, the lines $M\,N$, $C\,Q$, $F\,B$ are equal; and because $N\,E$ is parallel to $D\,A$, the lines $M\,N$, $P\,C$, $A\,E$ are equal. But because $Q\,F$ and $N\,E$ are parallel, $E\,F$ equals $C\,Q$; and $C\,Q$ has been shown to equal $M\,N$, therefore $E\,F$ equals $M\,N$. Now $F\,B$ and $A\,E$ have been shown each to equal $M\,N$, wherefore $A\,E$, $E\,F$, $F\,B$, the three divisional parts of the line $A\,B$ are equal, and the given straight line $A\,B$ has been divided into three equal parts as required.

Prop. IV. Problem.

To divide a given straight line into any required number of equal parts.

From one extremity of the given line draw a line at right angles, or at any convenient angle less than a right angle, to the given line. From the point in which the two lines meet

cut the second line at any convenient distance. From that point of section, with the same distance as radius, cut the line in a second point of section. Again, from the second, with the same distance, cut the line in a third point of section; and so on until the number of equal divisions are the same as the required number into which the given line is to be divided. Join the last divisional point in the second line to the extremity of the given line opposite to that at which the two lines meet; and from each successive divisional point in the second line draw a line, parallel to the first line of junction, to intercept the given line. The successive points of interception will then divide the given line into the required number of equal parts.



Let the required number of parts be seven, and let $A\,B$ be the given straight line. From point B, at right angles to $A\,B$, draw a line of indefinite length. With centre B and any convenient distance $B\,a$, cut the line $B\,g$ in point a. With centre a and same distance cut the line in point b. And so on successively until the point g be the seventh point of section, and the line $B\,g$ is accordingly divided into seven equal sections. Join $g\,A$, and, from the successive points of section, *viz.*, f, e, d, c, b, a, draw lines parallel to $g\,A$, intercepting the given line $A\,B$ in the successive points 1, 2, 3, 4, 5, 6.

Because the successive triangles $6\,B\,a$, $5\,B\,b$, $4\,B\,c$, etc., are similar triangles and since the side $B\,a$ of the first angle by an addition equal to itself becomes the side $B\,b$ of the second, which by an addition equal to $B\,a$ becomes the side of the third triangle and so on; and because $B\,6$, $B\,5$, $B\,4$, etc., are also sides of the same successive similar triangles, the divisions of the given line $A\,B$ are equal each to each.

See *Plate* 2, at the end of Appendix to Part First, for some other modes of division.

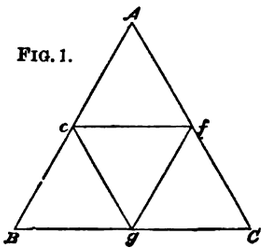
Fig. 2 shows a mode by which each divisional part of a line, divided into a certain number of equal parts, may be divided into the same number of equal parts. Fig. 3 is a modification which explains itself. In Fig. 4, from the six equal divisional parts of the line $A\,R$, six lesser equal divisional parts are cut off in the points b, c, d, e, f, g, by drawing the sines of the arcs. In Fig. 5 conversely, from the six equal divisions of $a\,g$, the six greater equal divisions (of $a\,r$) are obtained by drawing tangents to the arcs.

PROP. V. THEOREM.

Every triangle is divisible into any integral number of similar triangles equal each to each; and into any number, integral or fractional, of similar triangles.

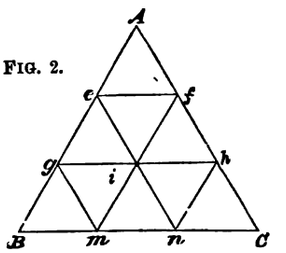
Let $A\ B\ C$ (fig. 1) be the triangle. Bisect the side $A\ B$ in point e, the side $A\ C$ in f, and the base $B\ C$ in g: join $e\ f$, $e\ g$, $f\ g$. Then, because point e bisects $A\ B$, and f bisects $A\ C$, the base $e\ f$ is parallel to the base $B\ C$, and the lesser triangle $A\ e\ f$ is similar to the greater triangle $A\ B\ C$. But therefore also the line $f\ g$ is parallel to $A\ B$, and $e\ g$ parallel to $A\ C$, consequently the other three lesser triangles $e\ B\ g$ $g\ e\ f$, $f\ g\ C$, are all similar to the greater triangle $A\ B\ C$. Now the base $e\ f$ is common to the two triangles $A\ e\ f$ and $g\ e\ f$; also $B\ g$ and $g\ C$ are each equal to $e\ f$; wherefore the four lesser triangles are equal each to each.

FIG. 1.



Again; let $A\ B\ C$ (fig. 2) be the triangle. Divide the sides $A\ B$, $A\ C$, $B\ C$, each into three equal parts in the points e and f, g and h, m and n, respectively; join $e\ f$, $g\ h$, $e\ n$, $f\ m$, intersecting in the point i, and join also $g\ m$, $h\ n$. Then, by parity of reasoning, it is evident that the nine lesser triangles $A\ e\ f$, $i\ e\ f$, $e\ g\ i$, $f\ i\ h$, $g\ m\ B$, $m\ g\ i$, $i\ m\ n$, $n\ i\ h$, $h\ n\ C$, are all similar to the greater triangle $A\ B\ C$, and are equal each to each. In like manner it may be shown that by dividing similarly the three sides of a triangle, each into the same number of equal parts as the other two, and joining the divisional points, the triangle will be divided into triangles similar to itself, equal each to each, and equal in number to the product of the number of the divisional parts of each of the sides multiplied by itself.*

FIG. 2.



Ideal Corollary.—Hence, because the *ultimate* divisional parts of a line are *ultimate* lines, and as such are indivisible (*Def.* 6), it follows that the *ultimate* triangle is also indivisible (*as a triangle*); that is to say, it is separable only into three ultimate lines.

SCHOLIUM.—The foregoing Theorem may be used to illustrate and demonstrate the following arithmetical proposition. Let any integral number n, equal $m + 1$. Then

* If, although the three sides of the triangle are similarly divided into the same number of respondently equal divisions, the divisional parts in each side are not all equal each to each, the lesser triangles, although similar, will evidently not be equal, each to each.

$(n + m) + (m + m\text{-}1) + (m\text{-}1 + m\text{-}2) + (m\text{-}2 + m\text{-}3)$ + &c. + $(n\text{-}m) = n^2$. For example, let $n = 10$. Then $19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1 = 10^2 = 100$.

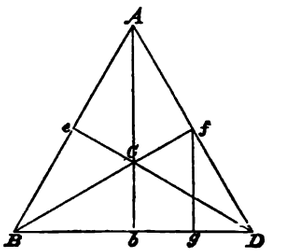
Scholium.—Since the greater eq. triangle $A\,B\,C$ is divided into nine lesser eq. triangles equal each to each, it is manifest on inspection that the eq. triangle $g\,A\,h$, of which the lineal magnitude (*dimension*) is proportional to the lineal mag. of the eq. triangle $A\,B\,C$ in the ratio of two to three, has the areal ratio to (the area of) the eq. triangle $D\,A\,B$ of four to nine. It demonstrates therefore (by inspection) the theorem; *Similar triangles are to one another in the duplicate ratio of their respondent sides.* (Euclid, IV. 19.)

It also demonstrates that if from an eq. triangle a lesser eq. triangle having one-third the dimension (*i.e.*, one-ninth the areal mag.) be cut off, there remain an eq. triangle and a parallelogram areally equal. Thus, *e.g.*, triangle $e\,B\,n$ manifestly equals the parallelogram $f\,e\,n\,C$. But (by Fig. 1) if the lesser triangle have one-fourth the areal magnitude of the greater, there will remain an equilateral triangle equal to that cut off and a parallelogram having twice the areal magnitude thereof.

Prop. VI. Theorem.

The centre of an equilateral triangle cuts off from each of the three vertex lines which respectively bisect the triangle, the one-third part of the line next to that side of the triangle intercepted by it, so that the part between the centre and the vertex of the angle bisected by the line, is twice the magnitude of the part between the centre and the side intercepted by the line.

Let $A\,B\,D$ be the equilateral triangle bisected in each of its three vertex lines $A\,b$, $D\,e$, $B\,f$, secting each other in the vertex centre C. The part $b\,C$ must be one-third of the line $b\,A$, $e\,C$ the one-third of $e\,D$, $f\,C$ the one-third of $f\,B$.

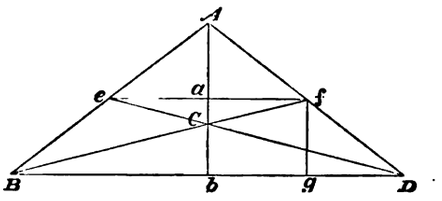


From point f, perpendicular to the side $B\,D$, draw $f\,g$. Then, because f bisects the side $A\,D$, point g bisects $b\,D$. But $b\,D$ is half of the side $B\,D$. Therefore $g\,b$ is one-third of $g\,B$, and $f\,C$ one-third of $f\,B$. And therefore also the part $e\,C$ is the one-third of $e\,D$, and the part $b\,C$ one-third of $b\,A$.

Corollary. Hence if a circle be inscribed in an equilateral triangle and a circle be circumscribed about the same triangle, the radius of the outer circle is twice the magnitude of the radius of the inner circle.—See the diagram to Prop. 10.

PROP. VII. THEOREM.

In an isosceles triangle, if each of the sides be bisected and from each point of bisection a line be drawn to the vertex of the opposite angle, the lateral centre of the triangle cuts off the one-third part of each of the three vertex lines at the end opposite the angle sected by the line.

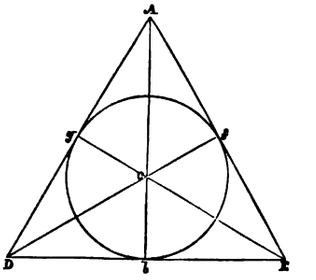


Let $A\,B\,D$ be the isosceles triangle, having the sides $A\,B$ and $A\,D$ equal. Bisect the sides $A\,B$ and $A\,D$ in the points e and f, and the base $B\,D$ in the point b. From the vertices A, B, D, of the three angles to the points bisecting the sides respectively, draw the lines $A\,b$, $D\,e$, $B\,f$, intersecting in the point C (the lateral centre of the triangle). The part $b\,C$ must be one-third of the line $b\,A$, $e\,C$ the one-third of $c\,D$, $f\,C$ the one-third of $f\,B$. Join $e\,f$ cutting the line $A\,b$ in the point a, and from f draw $f\,g$ perpendicular to $B\,D$. Because $f\,g$ is perpendicular to $B\,D$, and point g bisects $b\,D$, the part $f\,C$ is one-third of $f\,B$, and the part $e\,C$ one-third of $e\,D$ (*Prop.* 6). Again, because points f and e bisect the sides $A\,D$ and $A\,B$, the line $f\,e$ is at right angles to $A\,b$ and bisects $A\,b$ in point a. But $e\,C$ has been shown to be one-third of $e\,D$, and therefore $a\,C$ is one-third of $a\,b$. Now $A\,a$ is the half of $A\,b$, and when $a\,C$ (the one-sixth of $A\,b$) is added to it the sum $A\,C$ is two-thirds of $A\,b$, and the remainder $C\,b$ is one-third of $A\,b$. Wherefore the proposition of the theorem is demonstrated.

PROP. VIII. PROBLEM.

To inscribe a circle in an equilateral triangle.

Bisect one of the angles of the equilateral triangle, and from the line of bisection cut off the one-third next to the side of the triangle opposite the vertex of the bisected angle. With the point of section for a centre and the one-third of the line of bisection for a radius, describe a circle. The circle so described shall be the required circle inscribed in the given equilateral triangle.



Let $A\,D\,E$ be the equilateral triangle. Bisect the angle $D\,A\,E$ in the line $A\,b$. From $b\,A$ cut off one-third in the point C, and join $A\,C$, $D\,C$, $E\,C$. With centre C and radius $C\,b$ describe the circle $b\,f\,g$. The circle $b\,f\,g$ is the required circle inscribed in the equilateral triangle $A\,D\,E$.

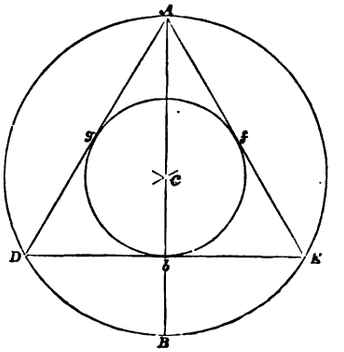
Because the triangle $A\,D\,E$ is equilateral, and, by Prop. 6, the point C is its vertex-centre, b, f, g, which are points in the sides of the triangle are also points in the circumference of the circle, and accordingly the circle $b\,f\,g$ is inscribed in the equilateral triangle $A\,D\,E$ as required.

(See the Scholium to Prop. 9 B.)

Prop. IX. (A). Problem.

To circumscribe a circle about an equilateral triangle.

Bisect one of the angles of the equilateral triangle, and from the line of bisection cut off the one-third next to the side of the triangle opposite the vertex of the bisected angle. With the point of section for a centre, and the two-thirds of the line of bisection for a radius, describe a circle. The circle so described shall be the required circle circumscribed about the given equilateral triangle.



Let $A\,D\,E$ be the equilateral triangle. Bisect the angle $D\,A\,E$ in the line $A\,b$. From $b\,A$ cut off one-third in the point C, and join $A\,C, D\,C, E\,C$. With centre C and radius $C\,A$ describe the circle $A\,D\,E$. The circle $A\,D\,E$ is circumscribed about the equilateral triangle $A\,D\,E$.

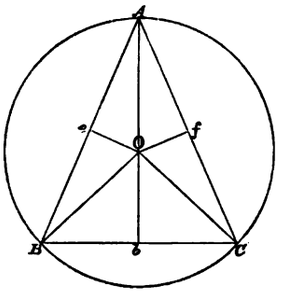
Because C is the centre of the equilateral triangle $A\,D\,E$ (*Def.* 19), and equidistant from the points A, D, E, (*Prop.* 1,) and because $C\,A$, $C\,D$, $C\,E$, are radii of the circle $A\,D\,E$, the circle $A\,D\,E$ is circumscribed about the triangle $A\,D\,E$.

Prop. IX. (B). Problem.

To circumscribe a circle about an isosceles or an equilateral triangle.

Bisect each of two equal sides of the triangle, and from each of the two points of bisection draw a line within the triangle at right angles to the side which the point bisects. With the point in which the two lines so drawn meet, and the distance from that point to the vertex of either of the three angles as a radius, describe a circle. The circle so described will necessarily pass also through the vertices of the other two angles, and be the circle circumscribing the triangle.

Let $A\,B\,C$ be any isosceles triangle. Bisect the base $B\,C$ in the point b, and join $A\,b$ bisecting the angle $B\,A\,C$, of which the arms $A\,B$ and $B\,C$ are equal. Bisect the sides $A\,B$ and $B\,C$ in points e and f respectively, and from e and f draw lines, at right angles to $A\,B$ and $B\,C$, meeting each other and the line $A\,b$ in point O. With centre O and distance $O\,A$, $O\,B$, or $O\,C$, describe a circle. The circle so described must be the circle $A\,B\,C$ circumscribing the triangle. Join $O\,B$ and $O\,C$.



Then because $f\,O$ is at right angles to $A\,C$ and f bisects $A\,C$, triangles $A\,f\,O$ and $C\,f\,O$ are similar and equal, and therefore side $O\,A$ of the one equals the respondent side $O\,C$ of the other. But the triangle $A\,O\,B$ is manifestly similar and equal to the triangle $A\,O\,C$, and consequently $O\,A$ also equals $O\,B$. Wherefore the point O is the vertex-centre of the isosceles triangle $A\,B\,C$; and the circle $A\,B\,C$, of which O is also the centre, circumscribes the triangle. (See *Def.* 20.)

For the case of the *equilateral* triangle see the diagram of the preceding problem.

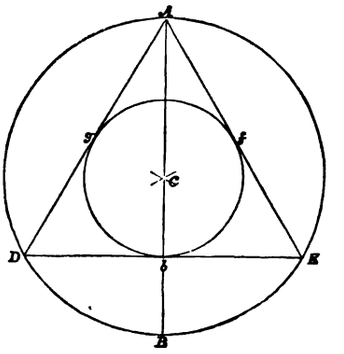
SCHOLIUM.—This and the *Proposition* preceding it repeat, in a measure, what has gone before, and are intended as illustrations. By reference to Prop. 1, the solutions become extremely simple, and the demonstrations manifest without formal statement. Thus:—*To inscribe the Circle;* With the vertex-centre of the triangle and the line joining the centre to the point bisecting one of its sides for radius describe a circle, which is manifestly the required circle inscribed in the triangle. *To circumscribe the Circle;* With the vertex-centre of the equilateral triangle and the line joining the centre to the vertex of one of the angles for radius describe a circle. The circle circumscribes the triangle, because the centre is equidistant from the three vertices (*Def.* 20). And, also: *A Circle* may be *inscribed in* or *circumscribed about an Isosceles Triangle,* in like manner, by taking the vertex-centre of the triangle (*Def.* 20) for the centre of the circle.

PROP. X. PROBLEM.

To inscribe an equilateral triangle in a given circle.

Find the centre of the given circle (Euclid, III. 1). Through the centre of the circle draw any diameter, and bisect one of the two radii into which the centre of the circle divides the diameter. Through the point in which the radius is bisected draw a line, at right angles to the radius, in both directions to the circumference of the circle. Join each of the two opposite points, in which the line so drawn intercepts

the circumference, to the point in the circumference at the furthest extremity of the diameter. The triangle so constructed shall be the required equilateral triangle inscribed in the circle.



Let $A\,D\,E$ be the given circle. Through the centre C draw a diameter $A\,B$. Bisect the radius $C\,B$ in the point b. Through b, at right angles to $A\,B$, draw $D\,E$, intercepting the circle's circumference in the opposite points D and E. Join $A\,D$ and $A\,E$. The triangle $A\,D\,E$ is the required equilateral triangle.

Join $C\,D$, $C\,E$, $D\,B$. Then, because $C\,b$ is half the radius $C\,B$, and at right angles to $D\,b$, the sides $C\,D$ and $B\,D$ are equal and the triangle $C\,D\,B$ is equilateral; therefore the angle $D\,C\,b$ contains two-thirds of a right angle. But $C\,b$ bisects the angle $D\,C\,E$; so that the angle $D\,C\,E$ contains two-thirds of two right angles, which is equivalent to one-third of four right angles; and, since the entire circumference of a circle subtends four right angles, the arc $E\,D$ is one-third of the circle's circumference. Now $b\,C$ is a part of $b\,A$, and consequently $b\,A$ bisects the arc $E\,A\,D$ in the point A. Therefore each half of the arc $E\,A\,D$, namely, the arc $A\,D$ and the arc $A\,E$, is one-third of the circle's circumference.

Wherefore the three chords $A\,D$, $D\,E$, $E\,A$, are equal, and the triangle $A\,D\,E$, inscribed in the given circle $A\,D\,E$, is shown to be equilateral.*

(Scholium.—It may be instructive to vary the demonstration as follows:—Join $C\,E$. Then, because $C\,E$ and $C\,A$ are both radii of the circle and therefore equal, the angles $C\,E\,A$ and $C\,A\,E$ are equal. Bisect $A\,E$ in the point f and join $C\,f$. Then, because the sides $A\,f$ and $E\,f$ are equal and $C\,f$ common, the triangles $f\,A\,C$ and $f\,E\,C$ are similar, and $C\,f$ is at right angles to $A\,E$. But $C\,b$ is at right angles to $D\,E$. Therefore the angles $C\,E\,b$, $C\,E\,f$, $C\,A\,f$, are equal each to each. Now

* It may be considered one of the blemishes in "Euclid's Elements" that no Construction is furnished therein whereby to directly inscribe an equilateral triangle in a circle. The problem, it is true, presents no difficulty, and some of Euclid's Constructions may be said to almost contain it. For example, by simply joining the alternate points in which the sides of the inscribed hexagon intercept the circle's circumference, the Construction to IV. 15, might be adapted to the purpose, and the demonstration be made direct and simple. As it stands, the student is referred to IV. 2, and finds he must describe an equilateral triangle outside the circle, and then fulfil the requisition of the problem by constructing within the circle a triangle equiangular to the triangle outside. When the primary character of each of the two figures—the rectilineal and the curvilineal—is considered, such an indirect method of procedure can scarcely be deemed satisfactory.

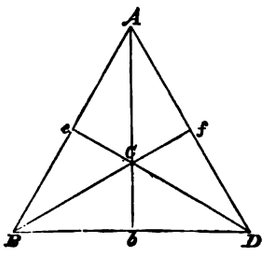
the angle CAf is the half of the angle EAD, and therefore the angles EAD and AED are equal. In like manner it may be shown that the angles EAD and ADE are equal. Wherefore the triangle ADE is shown to be equiangular.

COROLLARY.—Hence if with centre C and distance Cb a circle be described, the sides AD and AE, of the triangle, will touch that circle in their points of bisection g and f, and the circle $g\,b\,f$ will accordingly be *inscribed* in the triangle ADE.)

PROP. XI. PROBLEM.

With a given Straight Line to construct an equilateral triangle, such that the given straight line shall bisect the triangle.

Let Ab be the given straight line. From the line bA cut off one-third in the point C. (*Prop.* 3.) Through the point b at right angles to Ab draw a line in both directions, and with centre C and distance CA as radius cut the line so drawn in the points B and D. Join AB and AD. The triangle ABD so constructed shall be an equilateral triangle bisected by the line Ab.



Because Ab is perpendicular to BD, and the points B and D are equidistant from the point b, the line Ab bisects the triangle ABD, in the angle DAB.

From B and D, respectively, through C draw Bf and De. Then, by the Construction, CA, CB, CD are equal, and point C is the vertex-centre of the triangle. But also (by the Construction) Ce, Cf, Cb are equal. Wherefore, by Prop. 6, the triangle is equilateral and it is bisected in the given straight line Ab.

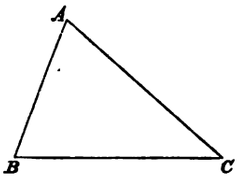
NOTE.—*Prop.* 4 of Euclid's Book I. seems to us, as it stands, objectionable in two respects. It is, as a whole, merely an elaborate and somewhat clumsy mode of assertion that the third sides of two triangles, each of which has one of its angles equal and *similar** to one of the angles in the other and two of its sides equal to the respondent sides in the other, must be equal, *because* they manifestly must be so. Moreover, in the endeavour to demonstrate this apparently superfluous *Theorem* (if it can be so called), the hypothesis of the third side being otherwise than straight is quite inadmissible; because a triangle has been strictly defined (*Euclid's Def.* 18) as "a plane figure contained by three *straight* lines." The following (Prop. 12) is suggested as a substitute: the Corollary being intended to supersede *Euclid's* I. 8 and I. 26.

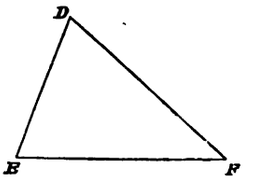
* *See Def.* 18.

PROP. XII. THEOREM.

If, in two triangles, one of the angles in one of them be equal to one of the angles in the other, and the two sides containing a second angle in the one be equal respectively to the respondent sides containing the respondent second angle in the other, the third side of the one must be equal to the third side of the other, and the other two angles in the one must be equal each to its respondent angle in the other triangle (and therefore the triangles must be similar and equal—i.e., *identical in form and equal in areal magnitude).*

In the triangles ABC, DEF, let the angle EDF be equal to the angle BAC, and the sides DE, EF equal respectively the respondent sides AB, BC; then must the third side DF be equal to the respondent third side AC, and the angles DEF, DFE, be equal and similar respectively to the respondent angles ABC, ACB.*





For, since the angles EDF, BAC, are equal, and the sides DE, EF, respectively equal to the sides AB, BC, the points D and F in the one, and the points A and C in the other, are equidistant, and therefore the lines DF and AC, which respectively connect these points, are equal, and the remaining two angles in the one are respectively equal to the remaining two angles in the other triangle. If not: let it be assumed that angles ACB and DFE are unequal. Then, since the sides EF and BC are equal, it would be impossible for the angles EDF and BAC to be equal, and by the construction they are equal.

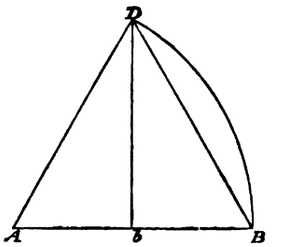
COROLLARY.—*If, in two triangles, two of the angles and one of the sides (whether the side between the angles or opposite to one of them) be equal respectively to two of the angles and one of the sides in the other, the angles and the sides are necessarily respondent, and the two triangles similar and equal*; because the point at the vertex of the third angle, in which the other two sides meet, must, in each triangle, be identically coincident in situation, relatively to the points at the extremities of the opposite side, with the point at the vertex of the third angle in the other, and therefore the third angles and the sides containing them must be equal.

* We here consider the case when one of the two equal sides is not one of the arms of the equal angle. For the case when the two equal sides are the two arms of the equal angles, see Prop. 15.

PROP. XIII. PROBLEM.

To describe an equilateral triangle upon a given straight line.*

Let $A\,B$ be the given straight line. Bisect $A\,B$ in point b; and from b draw a perpendicular to $A\,B$. With centre A and distance $A\,B$, describe an arc cutting the perpendicular in the point D. Join $A\,D$ and $B\,D$. The triangle $D\,A\,B$ shall be equilateral.

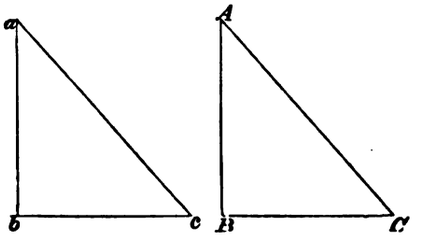


Because $A\,D$ and $A\,B$ are radii of the same circle $A\,D$ equals $A\,B$. But, because $A\,b$ equals $B\,b$, and $b\,D$ is perpendicular to $A\,B$, the two sides $b\,B$, $b\,D$, and the angle $B\,b\,D$ of the triangle $D\,B\,b$, are equal respectively to the two sides $A\,b$, $b\,D$ and the angle $A\,b\,D$ of the triangle $D\,A\,b$. Therefore the third side $B\,D$ of the one is equal to the third side $D\,A$ of the other. (Prop. 12.) Wherefore the three sides $A\,B$, $A\,D$, $B\,D$ are equal, each to each, and the triangle $D\,A\,B$ upon the line $A\,B$, is shown to be equilateral.

PROP. XIV. THEOREM.

If in two triangles one of the angles in one of them be similar† to one of the angles in the other, the two triangles are similar and each of the angles in the one is equal to its respondent angle in the other.

In the two triangles $A\,B\,C$, $a\,b\,c$, let the angle $B\,A\,C$ in the one be similar to the angle $b\,a\,c$ in the other. Then must the two triangles be similar, and the other two angles $A\,B\,C$, $B\,C\,A$, in the one must equal respectively the respondent angles $a\,b\,c$, $b\,c\,a$ in the other.



For, since (by *Def.* 18) the arm $A\,B$ of the angle $B\,A\,C$ has the same ratio to the

* It is scarcely necessary to remind the reader that this is the subject of Euclid's I. 1. The solution here given, the basis of which differs from that of Euclid, is intended to illustrate the theorem (*Prop.* 12) which precedes it. At the end of *Part Second* will be found a third mode of solution, illustrating the application of "Trisection."

† See *Def.* 18.

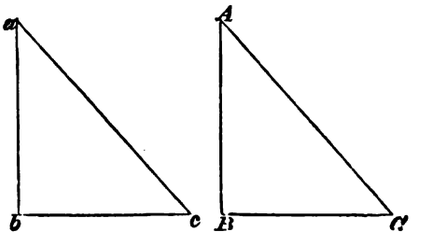
arm $A\,C$, which the arm $a\,b$ of the angle $b\,a\,c$ has to the arm $a\,c$, the ratio of the distance between the points B and C, at the extremities of the arms $A\,B$ and $A\,C$, to each of those arms respectively, must be the same as the ratio of the distance between the points b and c to the arms $a\,b$ and $a\,c$ respectively of the angle $b\,a\,c$. Therefore, when the lines, joining the points $B\,C$ and $b\,c$ respectively, are drawn, the angle $A\,B\,C$ must be equal to the respondent angle $a\,b\,c$, and the angle $A\,C\,B$ to the respondent angle $a\,c\,b$. Wherefore the two triangles are similar.*

Corollary.—If the distances between the extreme points of the similar angles be equal, or if one arm of the one be equal to the respondent arm of the other, then must *manifestly* the two triangles be equal as well as similar; for if not the two angles would not be *similar*. And, again, if in one of the triangles two of the angles be equal, and the distance between the extreme points of one of the angles be equal to one of its arms, that distance must be also equal to the other arm, and, when the lines joining the points in both angles are drawn, the two similar triangles must also be equilateral.

PROP. XV. THEOREM.

If in two triangles one of the angles in the one be equal to one of the angles in the other, and the arms of the equal angles be respondently equal, the third sides must also be equal; and the three angles in the one must be respondently equal to the three angles in the other triangle.†

In the two triangles $A\,B\,C, a\,b\,c$, let angle $A\,B\,C$ equal angle $a\,b\,c$, and the sides $A\,B$, $B\,C$, equal respondently the sides $a\,b$, $b\,c$. Then must the third side $B\,C$ equal the third side $b\,c$, and the three angles $A\,B\,C$, $B\,A\,C, A\,C\,B$, must equal respondently the three angles $a\,b\,c$, $b\,a\,c$, $a\,c\,b$.



For, since the angles $A\,B\,C$ and $a\,b\,c$ are equal, and the arms $A\,B$ and $B\,C$ respondently equal the arms $a\,b$ and $b\,c$, the points A and C in the one, and the

* The adoption of this basis for the demonstration would render proper the transference of the *def.* defining *ratio*, which in *Euclid* now precedes the Props. of Book V., to the commencement of Book I.

† In this Theorem the equality of the arms containing the equal angles is substituted for the equality of the ratios in Prop. 14. It is suggested that these Theorems and Corollaries (Props. 14 and 15) alone almost sufficiently cover the ground of Props. 4, 8, and 26 of *Euclid's First Book* and also of 4, 5, 6, and 7 of the *Sixth Book*.

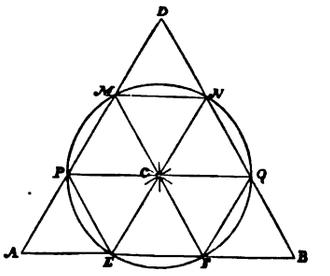
points a and c in the other are manifestly equidistant. Therefore the lines $A\,C$ and $a\,c$, respectively joining those points, are equal: and also the other angles $B\,A\,C$ and $B\,C\,A$ are respondently equal to the other angles $b\,a\,c$ and $b\,c\,a$, for, in each triangle, one of the arms of each angle is common to both, and the other arm of each is common to it and to that angle in the one triangle which (by the *Construction*) is equal to the respondent angle in the other triangle.

Corollary.—Hence also, by parity of reasoning, ***if two triangles have their three sides respondently equal, their three angles are respondently equal, and the two triangles are accordingly similar.*** For otherwise the points at the outer extremities of either two adjacent sides in the one triangle could not be equidistant with the points at the outer extremities of the respondent sides in the other triangle, and they *are* equidistant because the three sides of the two triangles are respondently equal.

Prop. XVI. Problem.

To inscribe a regular hexagon in a circle.

Let $M\,Q\,E$ be the circle. It is required to inscribe a regular equiangular hexagon in the circle $M\,Q\,E$.

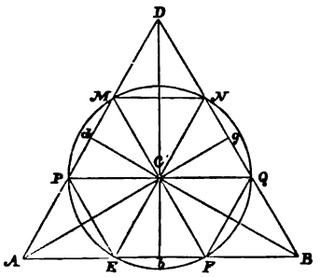


Through the centre C of the circle draw any diameter $P\,Q$. With centre Q and distance $Q\,C$ cut the circumference of the circle in points N and F; and with centre P and distance $P\,C$ cut the circumference in points M and E. Join each point in the circumference to the next, namely, M to N, N to Q, Q to F, F to E, E to P, P to M. The hexagon, so constructed, is a regular hexagon inscribed in the given circle, because the points are all in the circumference of the circle and the chords joining each point to the next are (*by the Construction*) equal each to each. For chords $E\,F$ and $M\,N$ are manifestly equal, and, because $M\,F$ is parallel to $N\,B$ and $N\,E$ parallel to $M\,A$, chord $M\,N$ is equal to radius $C\,Q$ and equal to radius $C\,P$. Consequently each of the six triangles within the circle is equilateral, and their outer sides are equal each to each. Wherefore the equiangular hexagon M, N, Q, F, E, P, is inscribed in the given circle as required.

PROP. XVII. PROBLEM.

With a given circle to describe an equilateral triangle such that the centre of the circle shall be also the vertex-centre of the equilateral triangle, and the three sides of the triangle shall be productions of the alternate sides of a regular hexagon inscribed in the circle.

In a given circle MQE inscribe the regular hexagon $MNQFEP$ (*Prop.* 16). Produce, in both directions, the alternate sides of the hexagon, namely, NQ, EF, PM, the productions meeting in the three points D, B, A. The triangle so constructed is an equilateral triangle, having its vertex-centre coincident with the centre of the given circle as required.



Draw, through the centre C of the circle from the vertices D, B, A, to the opposite sides of the triangle, the lines Db, Bd, Ag. Then because PQ is a diameter of the circle, CQ equals CP, and because NQ and MP are equal chords of the circle, the angle QDP is bisected in the line CD. Draw the diameters NE and MF. Then, because MF is a diameter of the circle and the chord FE equals the chord MP, the line CA bisects the angle FAM; and because NE is a diameter and NQ equals EF, the line CB bisects the angle NBE. It follows that the angles PDQ, MAF, NBE, are equal. Therefore the triangle is equiangular, and the point C, equidistant from the three vertices D, B, A, is the vertex centre. Now a part of each of the three sides of the triangle constitutes an alternate side of the regular hexagon inscribed in the circle. Wherefore the requisition of the problem is fulfilled.

Corollary.—Hence it follows that: ***If the alternate sides of a regular hexagon be produced in both directions, so that the produced lines meet in three points, the resulting triangle is equilateral.***

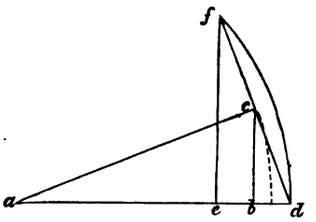
PROP. XVIII. PROBLEM.

To find a line (or lineal magnitude) which shall be a mean proportional between a greater and a lesser line of known magnitude.

From one extremity of the greater cut off a part equal to twice the difference between the two lines, and from the point of section draw a perpendicular of indefinite length. With the opposite extremity of the greater line as a centre of description and the line itself as a radius, describe an arc to cut the perpendicular, and, from the

point in which it cuts the perpendicular to the extremity of the greater line, draw the chord of the arc. Bisect the chord, and from the point of bisection draw a line to the centre of description (at the opposite extremity of the line). The line so drawn shall be the mean proportional between the given greater and lesser lines.

Let $a\,d$ equal the greater line and $a\,b$ the lesser. From the end terminated by d cut off $e\,d$ equal to twice $b\,d$, and from e draw the perpendicular $e\,f$ of indefinite length. With centre a and radius $a\,d$ describe the arc $d\,f$, cutting the perpendicular in f. Draw the chord $f\,d$ (of the arc), and bisect it in c; join $a\,c$. Then must the line $a\,c$ be the mean proportional between $a\,b$ the lesser and $a\,d$ the greater lines.



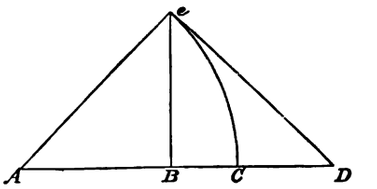
Join $b\,c$; then, because $e\,f$ is a perpendicular and c bisects $f\,d$ and b bisects $e\,d$, $b\,c$ is also a perpendicular to $a\,d$. And because $f\,d$ is the chord of the arc $d\,f$ and c bisects the chord, $a\,c$ is at right angles to $c\,d$. Hence by the similarity of triangles $a\,b$: $a\,c$:: $a\,c$: $a\,d$. Wherefore the line $a\,c$ is the mean proportional between the lines $a\,b$ and $a\,d$.

Prop. XIX. Problem.

*To find a third proportional to two given straight lines.**

Let $A\,B$ and $A\,C$ be the two given straight lines; it is required to find a third proportional to $A\,B$, $A\,C$.

In the greater line $A\,C$, take the part $A\,B$ equal to the lesser given line. From point B, perpendicularly to $A\,C$, draw a line, and with centre A and radius $A\,C$ cut that perpendicular in point e. Join $A\,e$. Produce $A\,C$ through C and from point e, at right angles to $A\,e$, draw $e\,D$ meeting the production of $A\,C$ in point D. Then, by similarity of the triangles, $A\,B$: $A\,C$ (or $A\,e$) :: $A\,C$: $A\,D$. Wherefore $A\,D$ is the third proportional to $A\,B$, $A\,C$.



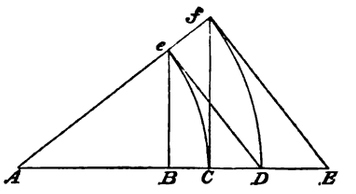
Prop. XX. Problem.

To three proportionals to find a fourth proportional; or to find a series of continued proportionals as many as required.

Let $A\,B$, $A\,C$, $A\,D$, be the three proportionals; it is required to find a fourth proportional to $A\,B$, $A\,C$, $A\,D$.

* This is also the subject of *Euclid's* VI. 11.

In the greater line $A\,D$ take a part $A\,C$ equal to the second line, and a part $A\,B$ equal to the least of the three given lines. From point B draw a perpendicular to $A\,D$, and, with centre A and radius $A\,C$, cut that perpendicular in point e. Join $A\,e$ and $e\,D$. Produce $A\,e$ through e, and, with centre A and radius $A\,D$ cut the produced line in point f. Produce $A\,D$ through D, and from f, at right angles to $A\,f$, draw $A\,E$ meeting the production of $A\,D$ in point E. Then, by similarity of triangles, $A\,B : A\,C :: A\,C : A\,D :: A\,D : A\,E$. Wherefore the line $A\,E$ is the required fourth proportional.



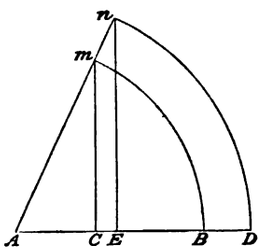
Evidently, by producing $A\,f$ through f, and with centre A and radius $A\,E$ cutting the produced line in point g, then producing $A\,E$ through E and from point g, at right angles to $A\,g$, drawing a line to meet the production of $A\,E$ in point F, the line $A\,F$ so found must be the fifth proportional; and, by proceeding in the same manner, any required number of successive proportionals may be found.

PROP. XXI. PROBLEM.

*To find a fourth proportional to three given straight lines.**

Let A, B, C, be the three given straight lines; it is required to find a fourth proportional to A, B, C.

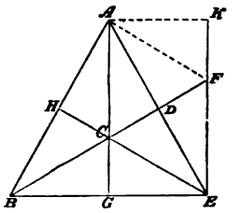
Take the straight line $A\,D$ equal to the given line C, and in it take the part $A\,B$ equal to the given line A, and the part $A\,C$ equal to the given line B. From point C draw a perpendicular to $A\,D$, and with centre A and radius $A\,B$ cut that perpendicular in point m. Join $A\,m$ and produce it through m, and with centre A and radius $A\,D$ cut the produced line in point n. From point n draw a perpendicular to $A\,D$ intercepting that line in point E. The line $A\,E$ is the fourth proportional: for, by similarity of the triangles, as $A\,B : A\,C :: A\,D : A\,E$. That is:—as the given line A is to the given line B, so is the third given line C to the line $A\,E$, which is therefore the required fourth proportional.



* This is also the subject of *Euclid's* VI. 12.

PROP. XXII. PROBLEM.

To describe a parallelogram areally equal to a given equilateral triangle, and such that half of the equilateral triangle shall be also half of the parallelogram.



Let $A\,B\,E$ be the given equilateral triangle. Bisect the side $B\,E$ in point G, and draw $A\,G$, bisecting the triangle. From point E, perpendicular to $B\,E$, draw $E\,K$ equal to $G\,A$, and join $K\,A$ parallel to $G\,E$. Then the side $A\,E$ of the given equilateral triangle is the diagonal of the parallelogram $A\,G\,E\,K$. Therefore the equilateral triangle is bisected in the line $A\,G$, and the parallelogram in the line $A\,E$, and the triangle $A\,G\,E$ is the common half of each. Consequently the given equilateral triangle and the parallelogram $A\,G\,E\,K$ are areally equal.

SCHOLIUM.—It will be seen that, in the Appendix to Part First, we object to the method of "Superposition." The diagram to the above may serve to illustrate a modified method of Superposition which is not objectionable. For example, let the line $A\,G$ be considered an immovable axis on which the triangle $A\,B\,G$ (of which it is one of the sides) can be turned over. It is readily demonstrable that, if so turned over, because $A\,G$ is at right angles to $B\,E$, and $G\,B$ and $G\,E$ are equal, the triangle $A\,B\,G$ will be geometrically applied upon the triangle $A\,E\,G$, the point B upon the point E, and each component part of the one upon the respondent part of the other. Again; suppose the side $A\,E$ the immovable axis and turn over the triangle $A\,G\,E$ on that lineal axis. The result must be fig. 2, which is a duplication of triangle $A\,G\,E$, and it is at once manifest that the figure $A\,G\,E\,H$ is also areally equal to the equilateral triangle $A\,B\,E$.

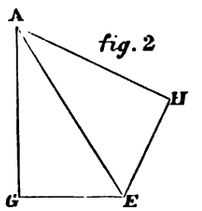


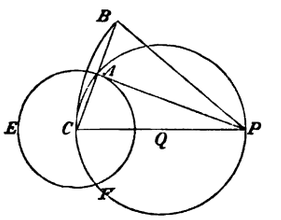
fig. 2

From the equality of the small triangles into which fig. 1 is divided, it is evident on inspection that the parallelogram $A\,C\,E\,F$ is areally equal to the four-sided triangular figure $A\,C\,E\,B$, because each figure contains four of the equal triangles.

PROP. XXIII. PROBLEM.

*From a given point outside the circumference, to draw a straight line which will touch the circumference.**

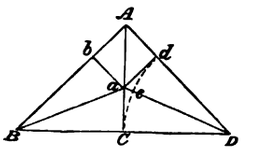
Let $A\,E\,F$ be the given circle, and P the given point outside the circle. Find C the centre of the circle, and join $C\,P$. Bisect $C\,P$ in Q. With centre Q and radius $Q\,C$ describe the circle $C\,A\,P\,F$, cutting the given circle in the points A and F. Join $P\,A$. The line $P\,A$ must touch the given circle at the point A. Produce $C\,A$ through A, and with centre P and radius $P\,C$ describe an arc cutting the production of $C\,A$ in B. Join $P\,B$. Then because radius $P\,C$ is twice radius $Q\,C$, and chord $C\,B$ is a production of chord $C\,A$, the greater arc $C\,B$ is *similar* to the lesser arc $C\,A$. The line $P\,A$ therefore bisects the chord $C\,B$ in the point A, and is at right angles to $C\,B$.—(*Euclid* III. 3.) But $C\,A$ is a radius of the given circle; and a line at right angles to a radius which meets the extremity of that radius in the circumference of the circle, touches that circle (*Euclid* III. 16.) Wherefore the line $P\,A$, drawn from the given point P, meets the radius $C\,A$ of the given circle at right angles in the point A, at the circumferential extremity thereof, and is tangent to the given circle.



PROP. XXIV. THEOREM.

If there be, within an isosceles right-angled triangle, an isosceles triangle having the same base and a vertical angle equal to three half-right angles, the right-angled triangle areally equals twice the triangle within it, together with a square, of which a line—joining the vertex of the inner triangle and the vertex of the right angle—is the diagonal.

Let $A\,B\,D$ be the isosceles right-angled triangle, of which $B\,A\,D$ is the right angle. Bisect the right angle in the line $A\,C$. With centre D and radius $D\,C$ describe the arc $C\,d$, intercepting $A\,D$ in d, and with centre B and radius $B\,C$, cut the side $A\,B$ in b. Bisect the arc $C\,d$ in e, and through c draw $D\,a$, intercepting $A\,C$ in a. Join $a\,B$, $a\,b$, $a\,d$. Then because the angles $a\,B\,C$ and $a\,D\,C$, at the base of the inner triangle, are together equal to half a right angle, the vertical angle $B\,a\,D$ equals three half-right angles. Because $a\,C$ is at right angles to $B\,D$, $a\,d$ at right angles to $A\,D$, and $a\,b$ at right



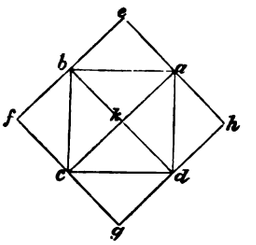
* This problem is also the subject of Euclid's III. 17.

angles to $A\,B$, the triangle $a\,C\,D$ equals the triangle $a\,d\,D$, and the triangle $a\,C\,B$ equals the triangle $a\,b\,B$. Now the square $A\,b\,a\,d$, of which $A\,a$ is the diagonal, added to the four equal triangles, completes the area of the right-angled triangle. Wherefore that right-angled triangle areally equals twice the inner triangle $a\,B\,D$ together with the square $A\,b\,a\,d$.

Prop. XXV. Theorem.

The square on the diagonal of a square has twice the area of the lesser square.

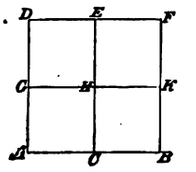
Let $a\,b\,c\,d$ be any square. The square on the diagonal ($a\,c$ or $b\,d$) must have twice the area of the square $a\,b\,c\,d$. Draw the diagonals $b\,d$ and $a\,c$, cutting each other in the central point k. Through points a and c, at right angles to $a\,c$, draw respectively the lines $e\,h$ and $f\,g$; and through points b and d, at right angles to $b\,d$, draw the lines $e\,f$ and $h\,g$, meeting the first two lines in the points $e\,f\,g\,h$. Then the square $e\,f\,g\,h$ is the square on the diagonal of the square $a\,b\,c\,d$, and manifestly each of the small squares $e\,b\,k\,a$, $b\,f\,c\,k$, $k\,c\,g\,d$, $a\,k\,d\,h$, is one-fourth of the square $e\,f\,g\,h$. Now the four small squares are bisected respectively by the lines $a\,b$, $b\,c$, $c\,d$, $a\,d$; and each of the halves is one-fourth of the given square $a\,b\,c\,d$. Wherefore a square has one-half the area of the square on its diagonal.



Prop. XXVI. (A). Theorem.

The square on half a line has one-fourth the area of the square on the whole line.

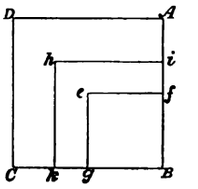
Let $A\,B$ be a given line, bisected in the point C. Upon $A\,C$ describe the square $A\,G\,H\,C$; and upon $A\,B$ describe the square $A\,D\,F\,B$. Bisect $D\,F$ in E, and $F\,B$ in K; and join $E\,H$ and $H\,K$. Then because points C, G, E, K, respectively bisect the sides $A\,B$, $A\,D$, $D\,F$, $F\,B$, of the greater square, the greater square contains four equal squares, of which the square $A\,G\,H\,C$, on the half-line $A\,C$ is one. Wherefore the square on the half-line $A\,C$ has one-fourth the area of the square on the whole line $A\,B$.



PROP. XXVI. (B). PROBLEM.

Within a given square to describe a square whose area equals half the area of the given square.

Let $A\ B\ C\ D$ be the given square. It is required to describe within $A\ B\ C\ D$ a square of one-half the area.

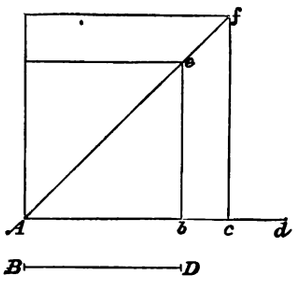


Bisect the side $A\ B$ of the given square in f, and on $f\ B$ describe the square $f\ e\ g\ B$. Then the area of the square $f\ e\ g\ B$ equals one-fourth the area of the given square (*Preceding Prop.*). Draw the diagonal $B\ e$ of the lesser square, and with centre B and radius $B\ e$ cut the side $A\ B$ of the given square in i, and the side $B\ C$ in k. Upon $B\ i$ describe the square $B\ i\ h\ k$. Then must the square $B\ i\ h\ k$ have an area equal to one-half that of the given square $A\ B\ C\ D$, because the square on the diagonal $B\ e$ has twice the area of the square $B\ f\ e\ g$ (*Prop.* 25), and the square $B\ f\ e\ g$ has one-fourth the area of the given square $A\ B\ C\ D$.

PROP. XXVII. THEOREM.

The square on a line has to the square on a second line greater or less than the first the duplicate ratio of the one line to the other.

Let the lines $A\ c$ and $B\ D$ be two unequal lines of which $A\ c$ is the greater. In the line $A\ c$ take $A\ b$ equal to $B\ D$, and find a third proportional $A\ d$ to the ratio of $A\ b$ to $A\ c$, so that as $A\ b : A\ c :: A\ c : A\ d$. (Prop. 19). Then the square on $A\ b$ must have to the square on $A\ c$ the same ratio as the line $A\ b$ to the line $A\ d$. Describe squares on the lines $A\ b$ and $A\ c$, and draw the diagonal $A\ f$ of the square on $A\ c$, including the diagonal $A\ e$ of the square on $A\ b$. Then because *similar triangles are to one another in the duplicate ratio of their homologous sides (Euclid* VI. 19) and the triangles $A\ e\ b$ and $A\ f\ c$ are similar, the triangle $A\ e\ b$ has the ratio to the triangle $A\ f\ c$ which the line $A\ b$ has to the line $A\ d$, and which is the duplicate ratio of the line $A\ b$ to the line $A\ c$. Now the triangles are the halves respectively of the squares on $A\ b$ and $A\ c$. Wherefore the square on the line $B\ D$



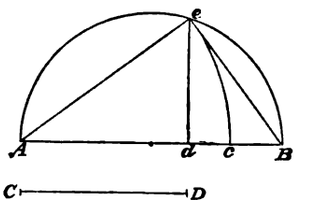
(equal to $A\,b$) has to the square on the line $A\,c$ the duplicate ratio of the line $B\,D$ to the line $A\,c$.

Corollary. Because *the diagonal of a square is a mean proportional in the ratio of the side of a square to a line twice as great as that side,* it follows from the above theorem, that *the square on the diagonal of a square has twice the areal magnitude of the lesser square.* (See Prop. 25.)

Prop. XXVIII. Theorem.

Of two unequal lines the square on the one has to the square on the other the duplicate ratio of the (corresponding) ratio which the one line has to the other (Prop. 27), and the square on a line which is a mean proportional between the other two lines has the same (simple) ratio to the square on either of the other lines respectively which the other two lines have (correspondingly) one to the other.

Let $A\,B$ and $C\,D$ be two unequal lines of which $A\,B$ is the greater. In the line $A\,B$ take a part $A\,d$ equal to the line $C\,D$. Find the lineal mean proportional $A\,c$ to the given lines $A\,B$ and $C\,D$, so that $A\,d\,(=C\,D) : A\,c :: A\,c : A\,B$ (*Prop.* 18). Then the square on $A\,d$ shall have the same ratio to the square on $A\,c$, and the square on $A\,c$ the same ratio to the square on $A\,B$ which the line $C\,D$ has to the line $A\,B$. For squares are one to the other in the duplicate ratio of their sides (*Prop.* 27), and therefore the square on $A\,c$ is in the duplicate ratio of $A\,d$ to $A\,c$. Now the duplicate ratio of $A\,d$ to $A\,c$ is the ratio of $A\,d$ to $A\,B$ (because $A\,c$ is the mean proportional of $A\,d$ to $A\,B$). Wherefore the square on the line $A\,c$ has the same ratio to the square on the line $A\,B$, and to the square on the line $C\,D$, respectively, which the line $C\,D$ has to the line $A\,B$, and the line $A\,B$ has to the line $C\,D$.

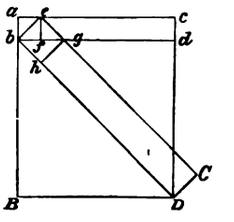


Prop. XXIX. Theorem.

The rectangle contained by the diagonal of the square on the greater and the diagonal of the square on the lesser, of the two lines which contain a given rectangle, has twice the area of that given rectangle.

Let $a\,d$ be the given rectangle, contained by the lines $a\,b$ the lesser and $b\,d$ the greater. The rectangle contained by the diagonal of the square on $a\,b$, and the diagonal of the square on $b\,d$, must have twice the area of the rectangle $a\,d$.

Draw $e\,f$ to complete the square on $a\,b$, and $e\,b$ the diagonal of that square. Also describe the square $b\,d\,D\,B$, on $b\,d$ and draw $b\,D$ the diagonal of that square. Complete the parallelo-

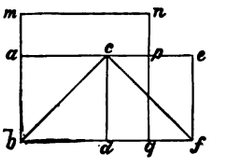


gram $e\,C\,D\,b$, and the square $e\,b\,h\,g$ on $e\,b$. Then, because the square on $e\,b$ has twice the area of the square on $a\,b$ (*Prop.* 25), and $b\,D$ has manifestly the same ratio to $e\,b$ that $b\,d$ has to $a\,b$, the greater rectangle $e\,D$ has twice the area of the given rectangle $a\,d$.*

PROP. XXX. (A.) THEOREM.

The area of the rectangle contained by the diagonal of a square and one side of the square is a mean proportional between the area of the square and the area of the square on the diagonal.

Let $a\,b\,d\,c$ be the square. Draw the diagonal $c\,b$, and from c at right angles to $b\,c$, draw $c\,f$. Produce $b\,d$ through d meeting $c\,f$ in the point f. With centre b and radius $b\,c$ cut $d\,f$ in q. Upon $b\,q$ describe the square $b\,m\,n\,q$, and upon $d\,f$ complete the square $d\,c\,e\,f$. Then must the rectangle $a\,q$, contained by $a\,b$ and $b\,q$, be a mean proportional between the square $a\,b\,d\,c$ and the square $b\,m\,n\,q$.†



Because $b\,c\,f$ is a right angle and $b\,d\,c$ also a right angle, $b\,d : b\,c :: b\,c$ (or $b\,q$) : $b\,f$, and since the altitude $d\,c$ is the same, $a\,b\,d\,c : a\,b\,q\,p :: a\,b\,q\,p : a\,b\,f\,e$. But the square on the diagonal of a square is areally equal to twice the lesser square (*Prop.* 25). Therefore the rectangle $a\,f$ (which manifestly equals twice the square $a\,b\,d\,c$) equals the square $m\,b\,q\,n$. Wherefore the rectangle $a\,q$, contained by the diagonal $b\,c$ and the side $a\,b$ of the given square, is a mean proportional between the given square $a\,b\,d\,c$ and the square $m\,b\,q\,n$, on the diagonal.

Corollary 1. Hence it follows (because rectangle $p\,f$ equals rectangle $p\,m$) that the rectangle contained by the side of a square and the *difference* between the diagonal and twice the side of the square,‡ equals the rectangle contained by the diagonal and the *difference* between the side of the square and the diagonal.

Corollary 2. And (for the same reason) it follows that the rectangle, contained by one side of a square and by a line greater than the diagonal in the same ratio that the diagonal is greater than the side of the square, equals the square on the diagonal.

Corollary 3. And it also follows, because rectle. $c\,q$: rectle. $p\,m$ (or $p\,f$) : : rectle. $a\,q$: sq. $m\,b\,q\,n$ (or rectle. $a\,f$) that the rectangle contained by the side of a square, and the

* It will be found that in the diagram the lesser rectangle contains seven of the lesser squares, and the greater rectangle seven of the greater squares.

† In the engraving, $b\,q$ and the square upon it are somewhat too great.

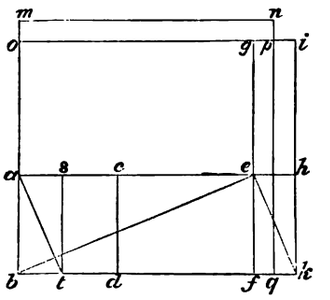
‡ Or, the *difference* between the diagonal of the first square and the diagonal of the square on the first diagonal.

difference between the side and the diagonal of the square has the same ratio to the rectangle contained by one side of the square on the diagonal and the *difference* between one side of the greater and one side of the lesser, as that of the rectangle compounded by the lesser square and the lesser rectangle to the greater square (or to the rectangle equal to the greater square).

PROP. XXX. (B). THEOREM.

The area of the rectangle contained by the greater side and the diagonal of a given rectangle is a mean proportional between the area of the square on the greater side and that of the square on the diagonal.

Let $a\,e\,f\,b$ be the given rectangle. Draw the diagonal $b\,e$. Produce $b\,f$ through f, and, with centre b and radius $b\,e$, cut the production of $b\,f$ in point q. Then must the rectangle contained by $b\,f$ and $b\,q$ be a mean proportional between the square on $b\,f$ and the square on $b\,q$.



From e, at right angles to $b\,e$, draw $e\,k$, meeting the production of $b\,f$ in point k. On $b\,f$ describe the square $o\,b\,f\,g$, and on $b\,q$ the square $m\,b\,q\,n$; also complete the rectangle $k\,b\,o\,i$, cutting the line $q\,n$ in point p. On $a\,b$ describe the square $a\,c\,d\,b$, and from a, through $e\,b$ at right angles, draw $a\,t$, and from t draw $t\,s$ parallel to $a\,b$. Then, because $a\,t$ is at right angles to $e\,b$, the rectangle $a\,t$ is similar to the given rectangle $a\,f$, and has the same ratio to the lesser square $a\,c\,d\,b$ as the given rectangle has to the square $o\,b\,f\,g$. But rectangle $e\,k$ equals rectangle $a\,t$, and therefore rectangle $g\,k$ equals the square $a\,c\,d\,b$. Since $b\,q : b\,f :: b\,k : b\,q$ (*Prop.* 18), the rectangle contained by $b\,q$ and $q\,f$ equals the rectangle contained by $b\,f$ and $q\,k$. Now rectangle $m\,p$ equals the rectangle contained by $b\,q$ and $q\,f$, and therefore the gnomon $m\,p\,f$ equals the rectangle $g\,k$. (*Preceding Prop.*). But, since the side $o\,b$ is common, rectangle $o\,f$: rectangle $o\,q :: o\,q : o\,k$,* and rectangle $o\,k$ equals the square $m\,n\,q\,b$. Wherefore the rectangle $o\,q$ is a mean proportional between the square on the greater side $o\,f$ of the given rectangle and the square on the diagonal $b\,e$ (or $b\,q$).

Corollary.—Hence, because the gnomon $m\,p\,f$ equals rectangle $g\,k$, which equals the lesser square $a\,c\,b\,d$, the *difference* between the square on the greater side and the square on the diagonal of a rectangle equals the square on the lesser side. (See Prop. 31.)

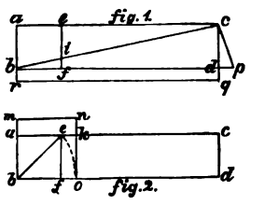
* See Prop. 34.

PROP. XXXI. THEOREM.

*The square on the diagonal of a rectangle exceeds the square on the greater of the two lines which contain the rectangle by the area of the square on the lesser of those two lines.**

Let $a\,c\,d\,b$ be the given rectangle; the square on the diagonal $c\,b$ must exceed the square on the line $b\,d$ by the area of the square on $a\,b$.

Describe on $a\,b$ the square $a\,e\,f\,b$ and join $c\,b$, cutting the side $e\,f$ of the square in the point i; produce $b\,d$ through d, and from c at right angles to $b\,c$ draw $c\,p$, cutting the produced line in p. Produce $c\,d$ through d and make $d\,q$ equal to $d\,p$. Draw $q\,r$ equal and parallel to $d\,b$ and join $b\,r$. Then, because $c\,p$ is at right angles to $b\,c$ and $c\,d$ equals $b\,f$, $f\,i$ equals $d\,p$, and since $b\,f$, $f\,e$, $a\,e$ are equal, $f\,i$ has manifestly the same ratio to $f\,e$ as $a\,e$ has to $a\,c$; therefore, since $d\,q$ equals $d\,p$, the rectangle $b\,q$ has the same ratio to the given rectangle $a\,d$ as the square $a\,b\,f\,e$ has to the given rectangle, and rectangle $b\,q$ equals square $a\,b\,f\,e$. Now because $b\,d\,c$ is a right angle and $b\,c\,p$ also a right angle $b\,d : b\,c :: b\,c : b\,p$, and therefore the rectangle contained by $b\,d$ and $d\,p$ equals the difference between the square on $b\,d$ and the square on $b\,c$. (*Prop.* 30A, *Cor.* 1). But the rectangle $b\,d\,d\,p$ is the rectangle $b\,q$, which has been shown to equal the square $a\,b\,f\,e$, wherefore the square on the lesser side, $a\,b$, of the given rectangle, equals the area by which the square on the diagonal $b\,c$ exceeds the square on the greater side, $b\,d$, of the given rectangle $a\,c\,d\,b$.



Corollary 1. Hence it follows that the rectangle contained by the greater side of a given rectangle, and the *difference* between the greater side and a line greater than the diagonal of the given rectangle in the same ratio as the diagonal is greater than the greater side, equals in area the square on the lesser side of the given rectangle. In Fig. 1 the rectangle $b\,q$ equals the square $a\,b\,f\,e$.

Corollary 2. Also because the gnomon $m\,k\,f$ (Fig. 2), equals the square $a\,b\,f\,e$ (*Prop.* 25), it follows that the difference between the square on the lesser side of a given rectangle and the square on the diagonal of that lesser square, equals the rectangle contained by the greater side of the given rectangle and the difference between that side and a line having the duplicate ratio to it of the given rectangle's diagonal.

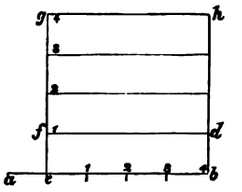
* It will be equivalent to state the Proposition thus:—*The square on the diagonal of a rectangle areally equals the sum of the squares on the two lines which contain the rectangle.*

PROP. XXXII. THEOREM.

If a straight line be unequally divided into two parts, such that the lesser is contained m times in the greater, the square on the greater part contains m times the rectangle contained by the two parts.

Let m equal four, and let $a\,b$, the given line, be divided in the point e, so that $e\,b$ equals four times $a\,e$. The square on $e\,b$ must equal four times the rectangle contained by $a\,e$ and $e\,b$.

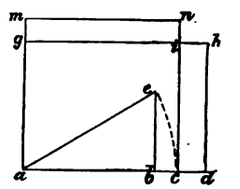
On $e\,b$ describe the square $e\,g\,h\,b$, and in the side $e\,g$ take $e\,f$ equal to $a\,e$. From f, parallel to $e\,b$, draw $f\,d$ intercepting the side $b\,h$ of the square in the point d. Then, because $e\,f$ equals $e\,a$ and $e\,g$ equals $e\,b$, the rectangle $f\,b$ is one-fourth of the square $e\,g\,h\,b$.



PROP. XXXIII. THEOREM.

If three lineal magnitudes be in continued proportion, so that the first is to the second as the second is to the third, the rectangle contained by the first and third equals the square on the second.

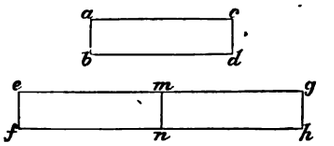
Let $a\,b$ and $a\,c$ be the first and second lineal magnitudes, of which $a\,c$ is greater than $a\,b$. On b draw a perpendicular to $a\,b$. Produce $a\,c$ through c. With centre a and radius $a\,c$ cut the perpendicular in the point e, join $a\,e$, and from e at right angles to $a\,e$ draw $e\,d$ intercepting the production of $a\,c$ in d. Then, because $a\,e\,d$ is a right angle and $a\,b\,e$ also a right angle, $a\,b : a\,c :: a\,c : a\,d$, and therefore $a\,d$ is the third lineal magnitude. Upon $a\,c$ describe the square $a\,m\,n\,c$; on d, perpendicular to $a\,d$, draw $h\,d$ equal to $a\,b$, and from h, at right angles to $d\,h$, draw $h\,g$ intercepting the side $a\,m$ of the square in the point g and cutting the side $n\,c$ of the square in the point i. Then by Prop. 30, *Coroll.* 1, rectangle $m\,i$ equals rectangle $i\,d$, and rectangle $g\,d$ contained by the first and third lineal magnitudes equals the square $a\,m\,n\,c$ on the second.



PROP. XXXIV. THEOREM.

Equiangular parallelograms having two of the sides of the one equal to two of the sides of the other, and the other two sides of the one unequal to the other two sides of the other, are to each other in the (simple) ratio of their unequal sides.

Let *a c d b* and *e g h f* be the two parallelograms, in which the sides *a b* and *c d* of the one are equal to the sides *e f* and *g h* of the other, and the other sides *a c* and *b d* of the one are each one half of the sides *e g* and *f h* of the other. The ratio of the rectangle *a d* to the rectangle *e h* must be that of one to two.



Bisect the parallelogram *e g h f* in the line *m n*. Then, because the side *e g* of the greater is twice the magnitude of the side *a c* of the lesser, the two rectangles *e n* and *m h* are each areally equal to the rectangle *a d*. Therefore the whole parallelogram *e g h f* is areally equal to twice the parallelogram *a c d b*.

In the like manner it may be shown that, if the sides *a b* and *e f*, in any two equiangular parallelograms, be equal, whatever be the ratio of the side *b d*, of the one, to the side *f h*, of the other, the areal ratio of the one parallelogram to the other will be the same as the lineal ratio of those unequal sides.

PROP. XXXV. THEOREM.

The areas of rectilineal figures are proportional to the areas of similar rectilineal figures in the duplicate ratio of their respondent sides. Similar circles are areally proportional in the duplicate ratio of their diameters; and the areas of similar lateral curvilineal figures are proportional in the ratio of the similar rectilineal figures contained in them.

Fig 1.—Divide the side *a d*, of the parallelogram *a d o g*, into three equal parts in the points of equal division *b* and *c*; and divide the side *a g* also into three equal parts in the points *e* and *f*. From the points *e* and *f* draw lines parallel to *a d*, and from points *b* and *c* lines parallel to *a g*. It is manifest, on inspection of the figure, that the greater parallelogram contains nine parallelograms similar to itself and equal each to each, of which the lesser parallelogram *a b m e* is one, and four are contained in the intermediate parallelogram *a c n f*.

Then, because the duplicate ratio of *one* to *two* is *one* to *four*, and parallelogram *a b m e*, is contained four times in parallelogram *a c n f*, it is evident that the areas of the two parallelograms are in the duplicate ratio of their respondent sides *a b* and *a c* (or *a e* and *a f*). And likewise, because the duplicate ratio of *two* to *three* is *four* to *nine*, and parallelograms *a c n f* and *a d o g* contain respectively four times and nine times the lesser parallelogram *a b m e*, and the side *a c* is to side *a d* as two to three, it is apparent that the parallelograms are one to the other in the duplicate ratio of their respondent sides.

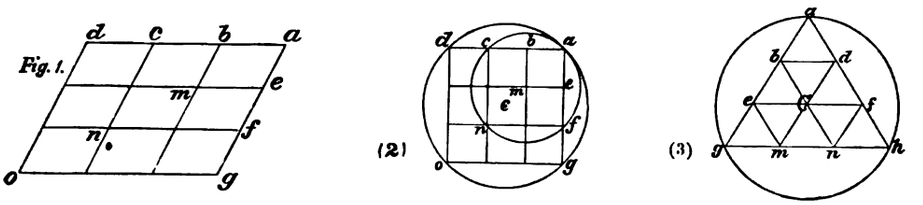


Fig. 2.—In like manner the areas of the squares *a e m b* and *a f n c* are proportional in the duplicate ratio of their respective sides *a e* and *a f*. With centre *m*, the vertex-centre of the lesser square, circumscribe the circle *a f n c*, and with centre *C*, the vertex-centre of the greater square, circumscribe the circle *a g o d*. Then, because radius *a m* is to radius *a C* as *one* to *one and a-half* and the diameters are as *two* to *three*, and the lesser and greater circles contain respectively the lesser and greater squares, it follows that their areas are in the duplicate ratio of their diameters. Moreover, the four segments of each circle may be considered as together constituting a quadrilateral curvilineal figure, and, as already shown, the curvilineal figures are in the same ratio as the squares contained in them.

Fig. 3.—As in the case of the parallelogram and of the square, each of the sides of the triangle *a g h* is divided into three equal parts, and, as before, it is apparent by inspection that the areas of the similar triangles are in the duplicate ratio of their respondent sides. With vertex-centre *c* and radius *c a* circumscribe a circle about the triangle *a e f*,* and with vertex-centre *C* and radius *C a* circumscribe a circle about the greater triangle *a g h*. Then, as before, it is apparent that the circles are in the same ratio as the triangles inscribed in them, and, since the three segments of each of the circles, of which the sides of each triangle are the chords, may be considered as together constituting a trilateral curvilineal figure, it is also again shown that similar lateral curvilineal figures are to each other in the ratio of the similar rectilineal figures contained in them.

* The Engraver has omitted this lesser circle in the diagram.

PROP. XXXVI. PROBLEM.

To describe a similar parallelogram, a square and a circle respectively twice the magnitude of a given parallelogram, square, and circle.

Fig. 1.—Let $A\,b\,c\,d$ be the given parallelogram. Produce the side $A\,b$ making $b\,e$ equal to $A\,b$. Find the mean proportional $A\,B$ to the ratio of $A\,b$ to $A\,e$, so that $A\,b : A\,B :: A\,B : A\,e$. (*Prop.* 18.)* Produce the side $A\,d$ making $d\,f$ equal to $A\,d$. Find the mean proportional $A\,D$ to the ratio of $A\,d$ to $A\,f$ so that $A\,d : A\,D :: A\,D : A\,f$. Draw $D\,C$ parallel to $A\,e$, and $B\,C$ parallel to $A\,f$. Then, because $A\,e$ and $A\,f$ are respectively twice the magnitude of $A\,b$ and $A\,d$, and $A\,B$, $A\,D$ are respectively the mean proportionals, the area of parallelogram $A\,B\,C\,D$ is twice the area of the given parallelogram $A\,b\,c\,d$. (*Prop.* 29.)

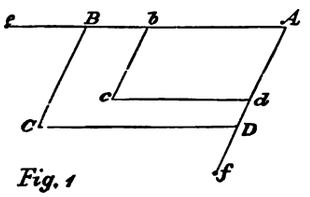
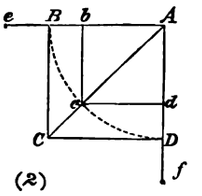
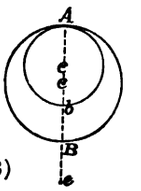


Fig. 1



(2)



(3)

Fig. 2.—Let $A\,b\,c\,d$ be the given square. Produce $A\,b$ making $b\,e$ equal to $A\,b$ and produce $A\,d$ making $d\,f$ equal to $A\,d$. With centre A and distance $A\,c$ cut $A\,e$ in point B and $A\,f$ in point D. Draw $D\,C$ parallel to $A\,e$ and $B\,C$ parallel to $A\,f$. Then $A\,B$ and $A\,D$ are respectively mean proportionals to the ratios of $A\,b$ to $A\,e$ and of $A\,d$ to $A\,f$, and $A\,e$ is twice the magnitude of $A\,b$ and $A\,f$ twice the magnitude of $A\,d$. Therefore the area of square $A\,B\,C\,D$ is twice that of square $A\,b\,c\,d$. (*Prop.* 25.)

Fig. 3.—Let $A\,b$ be the given circle. Through the centre c draw a diameter $A\,b$, and through b produce it, making $b\,e$ equal to $A\,b$. Find the mean proportional $A\,B$ to the ratio of $A\,b$ to $A\,e$ (*Prop.* 19). Bisect line $A\,B$ in point C, and with centre C and radius $C\,A$ describe the circle $A\,B$. Then, because line $A\,e$ is twice the magnitude of diameter $A\,b$ and diameter $A\,B$ is a mean proportional between $A\,b$ and $A\,e$, the area of the circle $A\,B$ is twice the area of the given circle $A\,e\,b\,f$. (*Prop.* 35.)

* Or thus: Describe a square on $A\,b$. Draw the diagonal from A to the point of the opposite angle. With centre A and the diagonal for radius cut the line $A\,e$ in the point B. For the lesser side: Draw from point A a line at right angles to $A\,e$, equal to $A\,d$. On the line so drawn describe a square, and, with its diagonal for radius and point A for centre, cut the line $A\,f$ in point D.

PROP. XXXVII. THEOREM.

Similar trilateral figures, having one, two, or all three of their sides curvilineal, are one to the other in the same ratio as the triangles contained in them, and accordingly are in the duplicate ratio of their respondent sides.

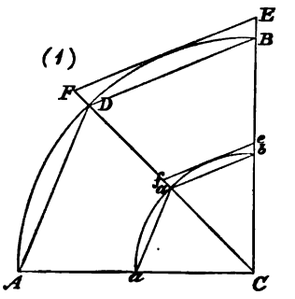


Fig. 1.—Let the octantal sector $C\,D\,B$, of which $D\,B$ is the arc and $C\,D$, $C\,B$ the two radii, be any sector of a circle. Bisect radius $C\,D$ in point d and with centre C and radius $C\,d$ describe the lesser arc $d\,b$. Draw the chords $D\,B$ and $d\,b$ of the two arcs. Produce the lines $C\,D$ and $C\,B$. Bisect the arcs and, tangent to them at the points of bisection, draw the lines $E\,F$ and $e\,f$.

Then, by similarity of the triangles, the ratio of triangle $F\,C\,E$ to triangle $f\,C\,e$ is the same as the ratio of triangle $B\,C\,D$ to triangle $b\,C\,d$. Triangle $F\,C\,E$ is manifestly greater and triangle $D\,C\,B$ less than the trilateral figure (*sector*) of which the arc $D\,B$ is the outer side. Likewise triangle $f\,C\,e$ is greater and triangle $b\,C\,d$ less than the trilateral figure (*sector*) of which the arc $d\,b$ is the outer side. And, because the triangles are similar, the *difference* $F\,E\,D\,B$, between the two greater triangles, has the same ratio to the difference $f\,e\,d\,b$, between the two lesser triangles, as the greater triangle $B\,C\,D$ has to the lesser triangle $b\,C\,d$. Now the *difference* between the triangles includes, on the interior side of the arc, the segment, and, on the exterior side, the area contained between the arc and the tangent line; and, since the segments are similar and the areas similar, it follows that the greater and lesser segments and also the greater and lesser areas have respectively the same ratio one to the other that the greater and lesser triangles have. Therefore the greater trilateral figure $D\,C\,B$, compounded of the triangle and segment, has to the lesser trilateral figure $d\,C\,b$, compounded of the triangle and segment, the same ratio as that of the triangles contained in them.

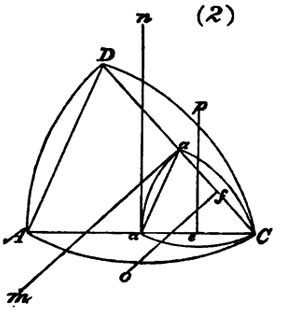


Fig. 2.—(Repeating *fig.* 1). From point d at right angles to $D\,C$ draw $d\,m$ equal to $C\,A$, and from point a at right angles to $A\,C$ draw $a\,n$ equal to $C\,A$. Bisect $a\,C$ in e and $d\,C$ in f. From point e at right angles to $a\,C$ draw $e\,p$ equal to $a\,C$, and from point f at right angles to $d\,C$ draw $f\,o$ equal to $a\,C$. With centres m, n, o, p respectively, describe the arcs $C\,D$, $C\,A$, $C\,d$, $C\,a$. Then must the trilateral figure $D\,C\,A$, having for each of its sides a cyclal curve—namely,

the arcs CA, CD, AD have to the similar trilateral figure dCa the same ratio that the triangle DCA contained in the greater has to the triangle dCa contained in the lesser. For the sides of the one trilateral figure are respondent to the sides of the other (because, by the *Construction* the curves are corresponding fractions of their lineal circles and therefore of equal curvature; *viz.*, two of the sides of the one are equal in curvature to two of the sides of the other, each to each, and the third side of the one is likewise equal and respondent to the third side of the other). Therefore what has been shown, with respect to the one curvilineal side (arcs AD and ad) of each, manifestly applies also to each of the other sides of each, and consequently applies to the whole of each figure. Accordingly the trilateral figure having two rectilineal sides CA, AD, and one curvilineal side CD, the trilateral figure having two curvilineal sides CA, CD, and one rectilineal side AD, and the trilateral figure having the three sides CA, CD, AD, all curvilineal, are each proportional respectively to each of the lesser similar figures in the same ratio as the greater triangle to the lesser. Now by *Euclid's* VI. 19, *similar triangles are one to the other in the duplicate ratio of their homologous sides.* Wherefore similar trilateral figures, having one, two, or all three of their sides curvilineal, are in the same ratio one to the other as the triangles contained in them, and in the duplicate ratio of their respondent sides.

Corollary 1.—It follows that all *sectors* and also all *segments* of circles are to the similar *sectors* or similar *segments* of other circles in the ratio of the triangles contained in them.

Corollary 2.—It follows, because squares and other quadrilateral rectilineal figures are respectively to other similar squares or other quadrilateral rectilineal figures in the same ratio as that of similar triangles and in the duplicate ratio of their homologous sides, that *quadrilateral figures, having two opposite curvilineal sides or having all four sides curvilineal, are to similar quadrilateral figures in the same ratio as that of the squares or other quadrilateral rectilineal figures contained in them, and in the duplicate ratio of their homologous sides.* (See *Prop.* 35.)

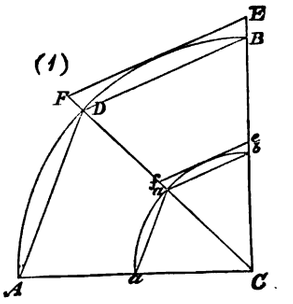
Scholium.—This Corollary applies likewise to similar polylateral figures having one or more respondent sides, or all of their sides, curvilineal.

PROP. XXXVIII. THEOREM.

Circles are to each other lineally as their diameters, and areally as the areas of the squares on their diameters.

(1.) The Lineal Ratio.—Let CA be a given radius. From C, perpendicular to CA, draw CB equal to CA. Bisect CA in a. With centre C and radius CA describe

the arc $A\,B$, and with centre C and radius $C\,a$ describe the arc $a\,b$. Bisect the arcs, $A\,B$ in D and $a\,b$ in d, and through point d join $C\,D$.



Because $C\,B$ is perpendicular to $C\,A$ the arcs $A\,B$ and $a\,b$ are quadrantal arcs and the half-arcs $A\,D$ and $a\,d$ are octantal arcs. Therefore the trilateral figure $C\,d\,a$ is similar to the greater trilateral figure $C\,D\,A$, and consequently the arc $A\,D$ is twice the magnitude of the arc $a\,d$ (because the side $C\,A$ is twice the mag. of $C\,a$).* Now each of the arcs is the eighth part of its circle, and the reasoning applied to the arc and its radius manifestly applies likewise to the entire circle and its diameter. Wherefore it is demonstrated that circles are to each other lineally as their diameters.

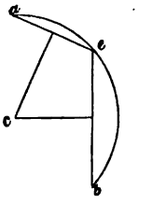
(2.) The Areal Ratio.—Draw the chords $A\,D$ and $a\,d$ of the arcs. Then, by *Prop.* 37, the two similar trilateral figures, each bounded on the outer side by the arc, are to each other areally as the triangles $C\,D\,A$ and $C\,d\,a$ contained in them; and, by *Prop.* 37, *Cor.* 1, similar sectors of circles are to each other in the ratio of the squares on the diameters of their circles. Since, therefore, areal circles are compounded of their sectors, it follows that circles have areally to each other the same ratio as the squares on their diameters.

Corollary.—Hence *areal circles are to one another* in the duplicate ratio of their diameters.

Prop. XXXIX. Problem.

To find the centre of description of a given arc.

Let $a\,b$ be the given arc. Divide the arc equally or unequally in the point e. Draw the chords $a\,e$ and $b\,e$ of the divisional arcs. Bisect each of the chords, and, from the point of bisection in each, draw a line at right angles to the chord. Then, because the parts of an arc are parts of the circumference of the circle to which the arc belongs, and because a line drawn from the centre to any chord in the circumference at right angles bisects the chord (Euclid III. 3), the point c in which the lines meet each other is the centre of description of the given arc $a\,b$; *i.e.*, the centre of the circle to which the given arc belongs.



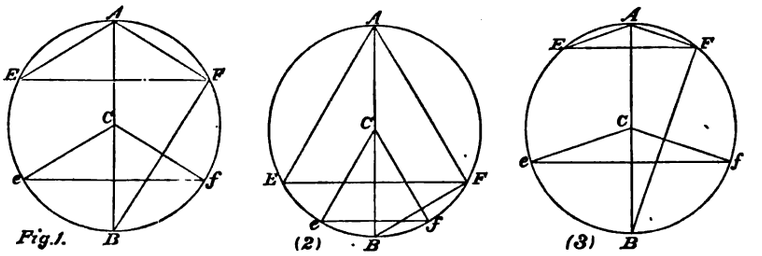
* Because radii $C\,A$ and $C\,D$ are twice the mag. of radii $C\,a$ and $C\,d$, the distance between the extreme points of $C\,A$ and $C\,D$ is twice as great as the distance between the extreme points of $C\,a$ and $C\,d$. Therefore, since the curves $A\,D$ and $a\,d$ are similar, the mag. of $A\,D$ is twice the mag. of $a\,d$.

PROP. XL. THEOREM.

If a triangle be described standing upon the circumference in the half of a circle with the vertex of the vertical angle at the centre of the circle, and the radius bisecting the vertical angle be produced through the centre of the circle to the opposite circumference, so that it becomes a diameter of the circle, and a second triangle, similar to the first, and also within the circle, be described, having the vertices of all three of its angles in the circumference, and having the point in the circumference intercepted by the produced radius at the vertex of its vertical angle, the arc of the circle cut off by the base of the second triangle will be twice as great as the arc cut off by the base of the first triangle.

Let $A\,E\,B\,F$ be the circle; within which the triangle $C\,e\,f$, standing upon the circumference at $e\,f$, has the vertex of its vertical angle at C. Produce the radius, $B\,C$, which bisects the vertical angle, through C to the opposite circumference at A; and describe a second triangle $A\,E\,F$, similar to the first triangle $C\,e\,f$, having its three vertices in the circumference at E, A, F, and having the vertex of its vertical angle at A. The arc $E\,B\,F$ cut off by the base of the second triangle must be twice as great as the arc $e\,B\,f$ cut off by the base of the first triangle.

Since the two triangles are similar, and the vertical angles of both are bisected by the same straight line, the corresponding (homologous) sides are parallel, and since the vertex of the vertical angle of the first triangle is the centre of the circle, and therefore bisects the diameter whose extremity meets the vertex of the vertical angle of the second triangle, it follows that if a chord $(F\,B)$ be drawn, joining the point (F) in the circumference intercepted by the base of the second triangle, with the point (B) in the circumference intercepted by the radius which bisects the first triangle, that chord must be bisected by the *radial* side $(C\,f)$ of the first triangle: for the side $(A\,F)$ of the second triangle is also a chord of the circle, and a radius bisecting that chord must be at right angles to it, and therefore the line $(F\,B)$, joining the extremity of that chord to the



terminal point (B) of the diameter which bisects both triangles, must also be at right angles to that chord,* and must consequently be bisected by the radial side (Cf) of the first triangle (which side is homologous, and parallel to the side of the second triangle). Now the two halves of a bisected arc are necessarily equal, wherefore, the arc cut off by interception of the base of the second similar triangle is twice as great as the (included) arc cut off by interception of the base of the first triangle.

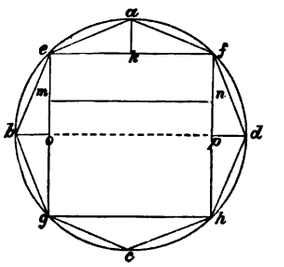
SCHOLIUM.—Hence: since, if an isosceles triangle, of which the vertical angle is a known fraction of two right angles, be so inscribed in a semi-circle that the centre of the circle is at the vertex of its vertical angle, a known fraction of the circumference of the semi-circle is cut off by the base thereof: it follows that, if a similar triangle be so inscribed in the complete circle as to have its vertical angle bisected by the produced radius, which bisects that of the first, the same fraction of the circumference of the entire circle will be cut off by the base of the second triangle. For example, (*conversely*) in Euclid's II. 4, where the base of the triangle cuts off one-fifth of the entire circumference, if a similar triangle be described in the half circle, standing on the circumference and having the centre of the circle at the vertex of its vertical angle, the arc subtending the triangle will be one-tenth of the entire circumference.

It may be observed that the base of the triangle, which has the extremity of the diameter at its vertex, is in no case twice as great as the base of that which has the centre of the circle at the vertex of its vertical angle; and it may be even less than the latter—namely, when the vertical angle of the former is obtuse and approaches to two right angles (*as in Fig.* 3).

PROP. XLI. PROBLEM.

To describe within a given regular octagon a parallelogram areally equal to half the given octagon.

Let $a\,b\,c\,d$ be the given regular octagon. In the octagon inscribe the square $e\,g\,h\,f$. Parallel to the side $e\,f$ of the square bisect the octagon in the diameter $b\,d$, cutting the sides $e\,g$ and $f\,h$ of the square in the points o and p respectively; bisect $e\,f$ in k, and from the vertex a of the triangle $a\,e\,f$ join $a\,k$. At the distances $o\,m$ and $p\,n$, each equal to $a\,k$, joining the points m and n in the sides $e\,g$ and $f\,h$ of the square, draw $m\,n$ parallel to $o\,p$. The parallelogram $m\,g\,h\,n$ must equal half the given octagon.



Because $o\,p$ equals $e\,f$, and $o\,m$ and $p\,n$ each equals

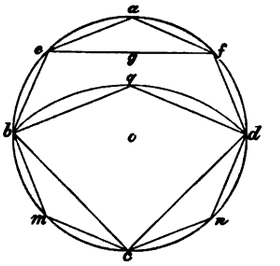
* Because $A\,F\,B$ is an angle in a semicircle, it is a right angle. The reasoning above shows why the two arms of an angle in a semicircle must be at right angles—namely, because, both arms being chords, if one of them be bisected by a radius the other arm must be parallel to that radius.

a k, the parallelogram *o m n p* equals twice the triangle *a e f*. Now the half octagon *b g c h d* contains the half-square *o g h p*, inscribed in it, together with the triangle *c g h*, equal to the triangle *a e f*, and the two half triangles *b o g* and *d p h* each equal to half the triangle *a e f*. Therefore the half octagon equals the half square *o g h p* and twice the triangle *a e f*, and the whole parallelogram *m g h n* equals in area the half of the given regular octagon.

PROP. XLII. PROBLEM.

To describe within a given regular octagon, two triangles together areally equal to half the octagon.

Let *a b c d* be the given regular octagon. Bisect the octagon by joining *b d;* join also *c b* and *c d*. With centre *c* and radius *c b* describe the arc *b q d*. Bisect the arc in the point *q*, and join *b q* and *d q*. The area of the two triangles *q b d* and *c b d* together must equal the area of half the given polygon. Join *e f*. Then, because the octagon is regular each of the two triangles *m b c* and *n d c*, equals the triangle *a e f*. And since the triangle *c b d* is inscribed in the semi-circle *b m n d*, the angle *b c d* is a right angle, and the sides *b c* and *d c* are equal, the arc *b q d* therefore is a quadrant. And because the arcs *b e, e a, a f, f d*, in the semi-circle *b a d*, are equal, the arc *e a f* is also a quadrant. Bisect the diameter *b d* in *o*. Now because the triangle *c b d* is isosceles and its base *b d* bisects the octagon and the angle *b c d* is a right angle, the line *c b* is the diagonal of a square of which one side is the radius *o b* (or *o c*) of the circle which circumscribes the octagon. Therefore, if *o e* and *o f* be joined, the half square (or triangle) *c b d* equals twice the half square (or triangle) *o e f* (Prop. 25). And accordingly the segment *b q d o* belonging to the greater triangle equals twice the segment *e a f g* belonging to the lesser triangle. But each of the two segments *c m b* and *c n d* equals the segment *e a f*, and it is manifest that the similar triangles in the similar segments are to each other as the segments which contain them. Consequently the triangle *q b d* equals the two triangles *m b c* and *n d c*. Wherefore the two triangles *b c d* and *b q d* together areally equal half the given octagon; and the trapezium *q b c d* is compounded of those two triangles.



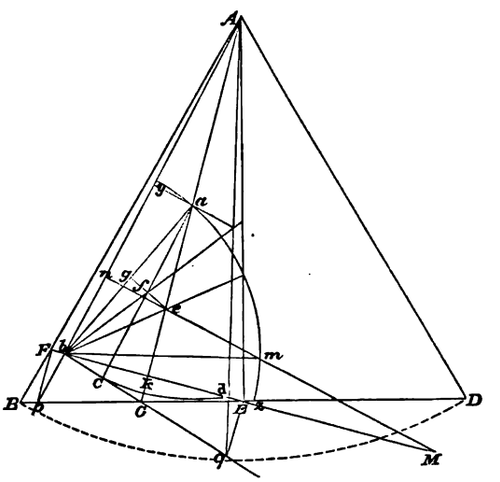
COROLLARY.—Hence it follows (if the two octagons be equal) that the quadrilateral figure *q b c d* equals the parallelogram *m n h g* of the preceding proposition, because each of them has been shewn to equal half of the regular octagon.

PROP. XLIII. PROBLEM.

To construct an isosceles triangle having each of the angles at the base treble that of the vertical angle.

Construct an equilateral triangle $A\,B\,D$ (*Euclid* I. 1). Bisect the angle $B\,A\,D$ in the line $A\,E$. Bisect also the angle $E\,A\,B$ in the line $A\,G$, intercepting line $B\,E\,D$ in point G; and from the point E, through $A\,G$, at right angles, draw $E\,F$ intercepting $A\,B$ in F. Join $F\,G$ and produce it indefinitely through G. From the points F and E, at right angles to the line $F\,E$, draw $F\,p$ and $E\,q$ intercepting respectively the line $B\,E$ and the production of $F\,G$ in the points p and q. Join $A\,p$ and $A\,q$ cutting the line $F\,E$ in the points b and d. The triangle $A\,b\,d$ shall be the required triangle having each of the angles $A\,b\,d$, $A\,d\,b$, at the base, treble the vertical angle $b\,A\,d$.

Bisect $A\,b$ in y, and, perpendicular to $A\,b$ draw $y\,a$ intercepting the line $A\,G$ in a. Join $a\,b$. Bisect $a\,b$ in g; and, perpendicular to $a\,b$, draw $g\,e$ intercepting $A\,G$ in e. Bisect the angle $b\,a\,e$ in the line $a\,f\,c$, cutting $g\,e$ in f. Join $f\,b$ and $e\,b$. Then:—(1.) Because the line $A\,G$ bisects the angle $b\,A\,d$, the angle $b\,A\,G$ is one-half the angle $b\,A\,d$. (2.) Because the point y bisects $A\,b$, and $y\,a$ is perpendicular to $A\,b$, the angle $a\,b\,y$ is equal to the angle $b\,A\,G$ (or $y\,A\,a$). (3.) Because the point g bisects $a\,b$, and $g\,e$ is perpendicular to $a\,b$, the angle $e\,b\,g$ is equal to the angle $e\,a\,g$. Also, because the line $a\,f\,c$ is parallel to $A\,b$, the angle $g\,a\,f$ equals the angle $y\,b\,a$ (which equals the angle $y\,A\,a$). But the line $a\,f\,c$ bisects the angle $g\,a\,e$, and since the line $g\,e$ (perpendicular to $a\,b$) bisects the angle $a\,e\,b$, the line $b\,f$ also bisects the angle $g\,b\,e$, and therefore the angle $f\,b\,e$ equals the angle $f\,b\,g$, which equals the angle $g\,b\,y$. Therefore each of the three angles $g\,b\,y$, $f\,b\,g$, $f\,b\,e$, equals the angle $b\,A\,G$. Produce the line $F\,E$ through E indefinitely; from the line $A\,b$, and at right angles to that line, through the point e, draw the line $n\,e\,M$, intercepting the production of the line $F\,E$ in the point M. Then: Because $A\,G$ is perpendicular to $b\,M$,* and $M\,n$ perpendicular to $A\,b$,



* $b\,M$ is a part of the line $F\,E\,M$, and $A\,G$ is perpendicular to $F\,E$.

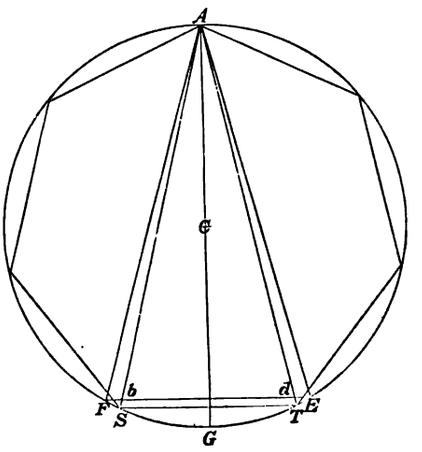
and $A\ G$ and $M\ n$ cut each other in the point e, the angle $b\ M\ n$ is equal to the angle $b\ A\ G$, and the triangle $b\ M\ n$ similar to the triangle $b\ A\ k$. Since, therefore, $b\ A$ is bisected in the point y, let the line $b\ M$ be bisected in the point z, and $z\ m$ perpendicular to $b\ E\ M$ be drawn intercepting $M\ n$ in m,* and $m\ b$ be joined, then the angle $m\ b\ d$ is manifestly equal to the angle $y\ b\ a$, and the angle $m\ b\ e$ to the angle $a\ b\ e$. Therefore the line $b\ e$ is shown to bisect the angle $A\ b\ d$, and, since the angle $A\ b\ e$ has been shown to contain three angles each equal to the angle $b\ A\ e$, the angle $e\ b\ d$ also contains three angles each equal to the same angle $b\ A\ e$. Now $b\ A\ e$ is half of the angle $b\ A\ d$, and consequently the angle $A\ b\ d$ contains three angles each equal to the angle $b\ A\ d$, and, since the two angles $A\ b\ d$ and $A\ d\ b$ at the base of the isosceles triangle $A\ b\ d$ are equal, each of these angles is three times the vertical angle $b\ A\ d$.

Wherefore an isosceles triangle $A\ b\ d$ has been constructed having each of the angles at the base treble the vertical angle as required.

PROP. XLIV. PROBLEM.

To inscribe a regular heptagon in a circle.

Let $A\ F\ E$ be the given circle. Draw any diameter $A\ G$, and, with $A\ G$ as the line of bisection, inscribe in the circle the triangle $A\ F\ E$ equiangular with the triangle $A\ F\ E$ of the preceding proposition, and within the triangle $A\ F\ E$, describe the triangle $A\ b\ d$ equiangular with the triangle $A\ b\ d$ of the preceding proposition. Through the points b and d respectively, to the circumference of the circle, draw the lines $A\ S$ and $A\ T$. Join $S\ T$, and inscribe in the circle the complete seven-sided polygon, having each of the other six sides equal to the side $S\ T$.



Then, because the triangle $A\ S\ T$ is equiangular with the triangle $A\ b\ d$ of the preceding proposition, each of the angles $A\ S\ T$ and $A\ T\ S$ at the base is treble the vertical angle $S\ A\ T$. The vertical angle $S\ A\ T$, therefore, is one-seventh of two right angles, and if inscribed in a semicircle (by making the angle point of the vertical angle

* The line $z\ m$ (respondent to $y\ a$) is omitted in the engraving. Also the point z (bisecting $b\ M$) should be nearer to the point E.

coincident with the centre of the circle) the arc intercepted between the arms of the angle standing on the circumference would be one-seventh of the semicircle. Now, by Prop. 40, when a similar triangle is inscribed in the circle, having the angle point of its vertical angle in the circumference instead of at the centre of the circle, the arc intercepted between the arms standing on the circumference is twice as great, and, therefore, the arc $S\,T$ is one-seventh of the circle's entire circumference. Wherefore, since each of the other six sides equals the side $S\,T$, the seven-sided polygon, inscribed in the given circle, is the regular heptagon required.

PROP. XLIV. PROBLEM.—REPEATED.

Plate 1. Fig. 1.

To inscribe a regular heptagon in a circle.

Let $E\,\mathfrak{X}\,Y\,Z$ be the given circle: it is required to inscribe a regular heptagon in the circle $E\,\mathfrak{X}\,Y\,Z$. Find the centre A of the circle (*Euclid* III. 3), and from the central point A draw any radius $A\,E$ intercepting the circumference of the circle in the point E. At right angles to the radius $A\,E$, and through the point E in the circumference of the circle, draw the line $B\,E\,D$ of indefinite length. From the extremity E cut off one-third of the radius $E\,A$ in the point C. With centre C and distance $C\,A$ cut the line $B\,E\,D$ in the opposite points B and D, and join $A\,B$ and $A\,D$ thus constructing an equilateral triangle $A\,B\,D$ bisected in the line $A\,E$ (Prop. 11). Bisect the angle $B\,A\,E$ in the line $A\,G$, and from the point E, at right angles to $A\,G$ draw $E\,F$, intercepting $A\,B$ in the point F. From the point F, perpendicular to $F\,E$, draw $F\,p$ intercepting the line $B\,D$ in the point p. Join $F\,G$, and produce it indefinitely through G. From the point E, at right angles to $F\,E$, draw $E\,q$ intercepting the production of $F\,G$ in the point q. From the point p in the line $B\,E\,D$, through A the centre of the circle, draw the line $p\,S\,b\,A\,Z$, cutting the circumference of the circle in the point S, intercepting the line $F\,E$ in the point b, and also intercepting the circle's opposite circumference in the point Z. From the point q in the line $F\,G\,q$ through A draw the line $q\,T\,d\,A\,Y$, cutting the circumference of the circle in the point T, and the line $F\,E$ in the point d, and also intercepting the circle's opposite circumference in the point Y. From the point Z, through the radius $A\,Y$, at right angles, draw the chord $Z\,\mathfrak{X}$ intercepting the circle's circumference in the point $\mathfrak{X}$. The chord $\mathfrak{X}\,Z$ shall be one of the equal sides of the regular heptagon inscribed in the given circle.

Because the vertical angle $B\,A\,E$ is the one-half of the vertical angle of the equilateral triangle $A\,B\,D$, and the lines $F\,p$ and $E\,q$ are perpendicular to the line

$F E$ (which is at right angles to the line of bisection $A G$), and because the lines $p A$ and $q A$ meet each other in the point A at the vertex of the vertical angle of the triangle, the isosceles triangle $b A d$ has each of the angles at its base, namely the angles $A b d$ and $A d b$, treble the vertical angle $b A d$ (Prop. 43). Now the line $b A$ is a part of the line $S A Z$, and the line $d A$ a part of the line $T Y$, wherefore the angles $S A T$ and $Z A Y$, standing upon the opposite circumferences of the circle respectively have the angles at their respective bases each treble the vertical angle. But the three angles in a triangle are together equal to two right angles; consequently, since the vertical angle $Z A Y$ of the triangle $A Y Z$ equals one-third of each of the other two angles $A Z Y$ and $A Y Z$, it is subtended by one-seventh of the semi-circumference of the circle, on the circumference of which it stands. Now, because the chord $Z \mathcal{X}$ cuts the radius $A Y$ at right angles, $A Y$ bisects the angle $\mathcal{X} A Z$, and therefore the angle $Y A \mathcal{X}$ equals the angle $Z A Y$. Hence, since the angle $Z A Y$ is subtended by one-seventh of the semicircle $Y T$, the double angle $Z A \mathcal{X}$ is subtended by one-seventh of the complete circle $E \mathcal{X} Y Z$.

Wherefore the chord $Z \mathcal{X}$ is shown to be one of the seven equal sides of a regular heptagon inscribed in the given circle as required.

NOTE.—With regard to the following problem (Prop. 45).—*To inscribe a regular Nonagon in a Circle*—we are about to commit an irregularity. For its demonstration is dependent upon demonstration of "trisection of the angle," and we are therefore obliged to refer *onwards* to Prop. II., Part Second. In other respects, however, the method here adopted will be found, we think, to possess an independent interest, and also from its connection with the preceding propositions on the Heptagon, its appearance in this place seems more suitable, especially as we purpose in Part Second to solve the same problem (together with others of a similar character) by direct application of "Trisection." (See *Part Second*, page 107).

PROP. XLV. PROBLEM.

To inscribe a regular nonagon in a circle.

Let $a k b t y$ be the given circle: it is required to inscribe a regular nonagon in the circle $a k b t y$.

Plate 1, Fig. 2. And also the Fig. on page 65.

Find the centre C of the circle, and through C draw any diameter $a b$. Bisect the radius $C a$ in the point d, and produce $C a$ through a, making $a A$, the increased length of $C a$, equal to $d a$. Produce also the radius $C b$ through b, making $b B$, the production of $C b$, equal to $a A$. With centre a and radius $A C$ describe the arc $c y$

intercepting the circumference of the circle in y. Join $C\,y$ and produce it through y. With centre A and radius $A\,C$ describe the arc $C\,x$, intercepting the production of $C\,y$ in the point x. Cut off one-third of the angle $x\,A\,C$ in the radial line $A\,m$. (Prop. 2, Part Second). Join $C\,m$, cutting the circumference of the given circle in the point g. At the opposite extremity of the diameter $a\,b$ proceed in the like manner, namely:—With centre b and distance $b\,C$ cut the circumference of the given circle in the point t. Produce $C\,t$ through t, and with centre B and distance $B\,C$ describe the arc $C\,v$ intercepting the production of $C\,t$ in the point v. Cut off one-third of the angle $v\,B\,C$ in the line $B\,n$, intercepting the arc $v\,n\,C$ in the point n. (Prop. 2, Part Second). Join $C\,n$, cutting the circumference of the given circle in the point h. With centre g and distance $g\,h$ cut the circumference of the circle in the point 1. With the same radius and centre 1, cut the circumference in the point 2. With same radius and centre 2, cut the circumference in point 3. With same radius and centre 3, cut the circumference in point 4. With same radius and centre 4, cut the circumference in point 5. With same radius and centre 5, cut the circumference in point 6. With same radius and centre 6, cut the circumference in point 7. Join the respective divisional points in the circumference each to the next, namely, 1 to 2, 2 to 3, 3 to 4, and so on. The polygon so inscribed shall be the regular nonagon of nine equal sides.

Because the chord $C\,y\,x$, of the arc $C\,m\,x$ is necessarily at right angles to the line of bisection of the angle $C\,A\,x$, and the chord $C\,g\,m$, of the arc $C\,m$, is at right angles to the line of bisection of the angle $C\,A\,m$, the angle contained by the two chords, namely, the angle $g\,C\,y$, is equal to half of the angle $m\,A\,x$. Now the triangle $a\,C\,y$ is manifestly equilateral, for since y and a are points in the circumference of the same circle, and y and C are points in the same arc, the radii $C\,y$, $C\,a$, $a\,y$ are equal; and, because the chord $C\,y$ is a part of the chord $C\,x$, the greater triangle $A\,C\,x$ is similar to the lesser triangle $a\,C\,y$, and therefore also equilateral. Consequently, since the three angles in a triangle together equal two right angles, the angle $C\,A\,x$ equals two-thirds of a right angle, and the angle $m\,a\,x$ (which, by Prop. 2, Part Second, is the one-third of angle $C\,A\,x$) equals two-ninths of a right angle. But the angle $g\,C\,y$, contained by the two chords $C\,m$ and $C\,x$, equals one-half the angle $m\,A\,x$, and equals therefore one-ninth of a right angle. Referring now to that part of the figure belonging to the opposite extremity B of the diameter $A\,B$:—Similarly and by parity of reasoning, the angle $h\,C\,t$, which by the construction is equal and similar to the angle $g\,C\,y$, equals one-ninth of a right angle. Now the triangle $y\,C\,t$ is an equilateral triangle, and equal to the triangle $a\,C\,y$, because each of the equal sides of each of the equilateral triangles $a\,C\,y$ and $b\,C\,t$ is a radius of the circle in the semi-circumference of which $a\,y$ and $b\,t$ are chords, and therefore the intermediate

Fig. 2.

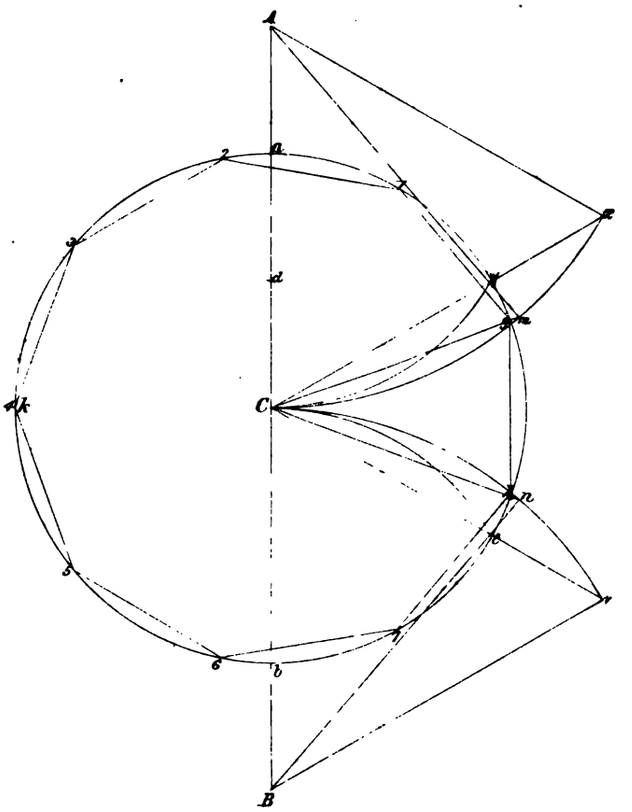
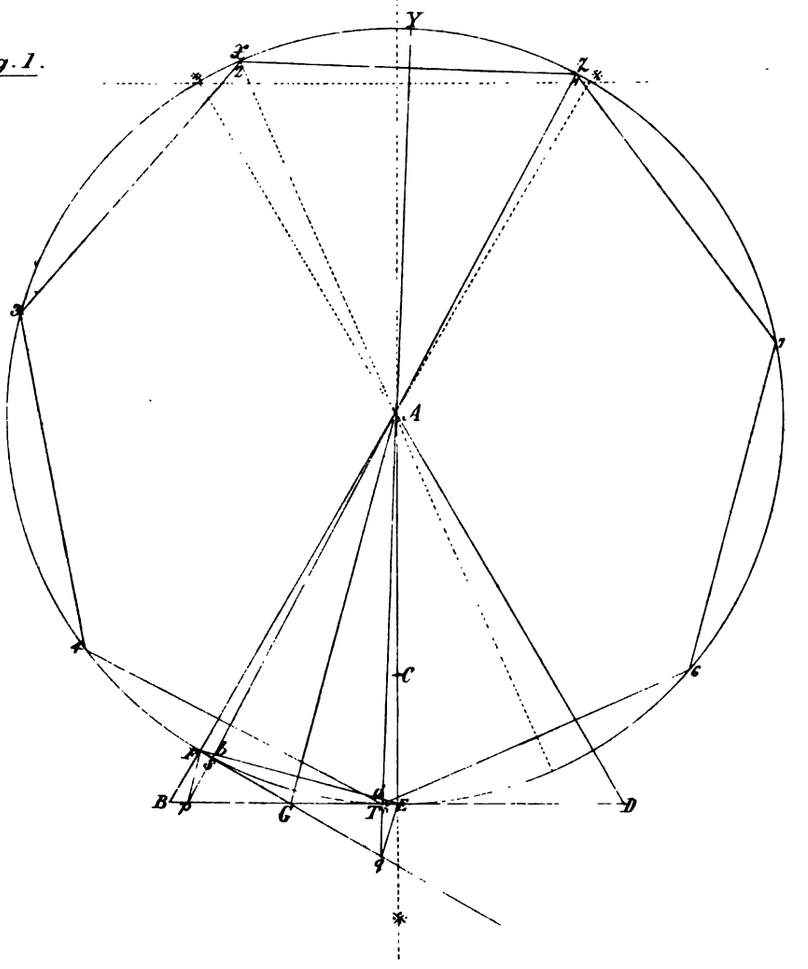
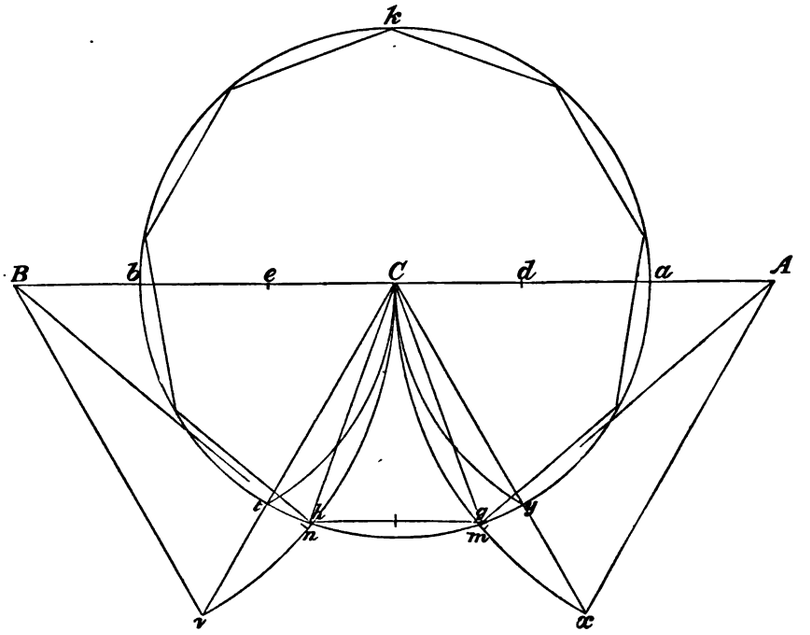


Fig. 1.



chord $y\,t$ also equals the circle's radius.* Consequently the angle $t\,C\,y$ equals six-ninths of a right angle. But each of the angles $y\,C\,g$ and $t\,C\,h$ equals one-ninth of a right angle, and when these are *together* taken from the angle $y\,C\,t$, the remaining angle $g\,C\,h$ equals four-ninths of a right angle. Nine times the angle $g\,C\,h$, therefore, equals four right angles, and, since a complete circle contains four right angles, the angle $g\,C\,h$ is contained nine times in the circle $a\,g\,h\,b\,k$.



Wherefore, since, by the construction, the other eight chords are made each equal to the chord $g\,h$, the nine-sided polygon is inscribed in the given circle and is shown to be a regular nonagon, as required by the problem.

* Because the semicircle contains the equivalent of two right angles, and as angles $a\,C\,y$ and $b\,C\,t$ each equals one-third of two right angles, the intermediate angle $y\,C\,t$ also equals one-third of two right angles.

APPENDIX TO PART FIRST.

APPENDIX TO PART FIRST.

EUCLID'S POSTULATES.—The sixth postulate is clearly a theorem and may be readily demonstrated. The two produced lines must meet, as postulated, because if the angles were both right angles, the two straight lines, being both perpendicular to the third line (*Euclid's Def.* 9), would be parallel, but as the interior angles are together less than two right angles it is manifest that, if the two straight lines be produced through the extremities opposite to the points in which they intercept the third line, they must meet.

Postulate the fifth.—"*That two straight lines cannot enclose a space*"—is an axiom; *i.e.*, a theorem of which the demonstration is inherent because manifest without formal argument or statement.

Postulate the fourth.—"*That all right angles are equal to one another*"— is a definition, or part of a definition. Whether it be considered imperfect or not, it simply defines (as far as it goes) what is to be understood by *equality* or *inequality* of angles.*

Postulates 1, 2, 3, stand by themselves and may perhaps be justified if a necessity or natural requisition by pure geometry for any *postulates* can be shown. By these the geometrician substantially asks to be allowed to use a pair of compasses (or their equivalent) to describe circles, and a sheet of paper and a pencil (or their equivalents) to delineate geometrical figures. Means and methods to which no reasonable mind would ever think of objecting if no formal request for them were made. But, although this is substantially the meaning, it was certainly not intended to ask for them *substantially*, that is to say—in a natural (*material*) sense. Euclid intends to ask that the imagination or the mind may be allowed to do these things, in, of course, an ideal sense. The first postulate, however, as it now reads in our manuals and treatises, asks "that a straight line may be *drawn* from one point to another." Euclid did not write his treatise in English or probably he would have used the word "extended." As it is, the Imagination might reasonably ask, in its turn, for fingers and a pencil (or something else) to draw with. Taking, however, the first postulate as a whole, there is a much more grave objection applying to it: for it may well suggest to the responsive

* It is indeed merely a repetition of a part of *Euclid's Def.* 9.

mind of the thoughtful student a reply to the following effect.—"You ask to be allowed to draw a straight line from point to point. I should not have the slightest objection if you had any *real* points. But you have strictly defined a point to be *no-thing;* and how can you possibly draw a line *from* no-thing *to* no-thing. As you put the responsibility upon me, I really cannot consent to your drawing a line under the circumstances." Evidently the dilemma would be a very awkward one.

Does Euclid himself, let us enquire, really care for these postulates or show any respectful regard for them if they, by their implied limitations, stand in the way of what he wishes to do? We turn to Prop. 4, of Book I. (not far on), and there find the right assumed to take up a triangle bodily from its place and to set it down on the top of another. The operation is supposed to be conducted very carefully, so that some one or more of the corresponding points and lines in the two figures coincide. But where is the postulative request for permission to do this?

The practical result of attempting to fence in the lawful territory of pure geometry with postulates, is the inference now commonly entertained, that *opinion* may be exercised to decide with regard to geometry what is desirable and what inexpedient; that experienced geometricians may lay their heads together and choose, for themselves and for others, what to allow and what to forbid; that the authority and example of this or that eminent geometrician may support and legitimatize a process or method of procedure. The method of Superposition has Euclid's example to begin with, and the circumstance of its being very handy and convenient for an additional recommendation. Just so: then why not adopt it to the full, and even carry its application further? Well: *this* is the reason:—Because it is an illegitimate mode of procedure. Illegitimate!! Why Euclid himself authorizes and makes use of it. Very true: but it is, notwithstanding, illegitimate.

To construct two figures, one superposed upon the other, either perfectly coincident or differing in some precisely specified manner, and then, the one figure remaining motionless (*fixed*), to rotate the other, upon a point common to both, throughout a definite arc of revolution, is legitimate. And, again, to turn over a figure, of which one part of the boundary is a straight line, upon the straight line as a fixed axis, is also legitimate. Because, in each case the requisite basis of certainty, which pure geometry imperatively requires, belongs to the procedure. But where is the certainty when a figure is taken wholly away from its primary locality and placed upon another figure or elsewhere? It is easy to say, apply the one to the other, and to *suppose* it applied accordingly: but where is the proof that is correctly applied?

It is very easy to assume that the one figure fits the other exactly, or to assume that it does not. But to *assume* either the positive or the negative is simply to "beg the question." If the two figures be constructively made exactly similar and equal, they are, of course, *by the Construction*, quite alike, and no proof is wanted. But if there be a possibility of doubt, let the perfect similarity and equality (or the contrary) be established and demonstrated in the proper manner. How can superposition remove the doubt? If the two figures are perfectly similar and equal, they will fit each other perfectly; if they are not alike they will not fit. The thing is to find out whether they really are alike. But to measure them with a tape line is not the correct way; although, of the two ways, it might be preferable to the method of superposition.

Pure Geometry is not arbitrary nor capricious, and does not want postulates. Any thing (of a suitable nature) may come into Her Domain and be examined, and, if found to be perfectly truthful, sound, law-abiding, and useful, is welcome to remain permanently. But any thing that makes a lie or has an inherent propensity for saying what is untrue—whether it be a statement, an argument, a method, or what else, is, as soon as it is found out, ordered off, without respect for persons, and *must go:* so must also any thing found to have secreted anywhere about its person the elements of *falsity* or of *uncertainty*.

EUCLID'S ELEMENTS.—Impressed, as we are, not only with the intrinsic value of Euclid's great work, but also with the advantage of having one universally recognised authority on the subject of which it treats, there are, we think, some very weighty objections applicable to the "Elements" which have not been hitherto stated; and, as Part First of the present treatise has partly for its object the correction and amendment of certain supposed defects in that work, we will here endeavour to point out, or at least to indicate the nature of, the supposed objections to which we have just referred.

For educational purposes, at the present time, the "Elements" seems to be made use of with the same sort of duplex indistinct purpose which appears to have pervaded its original composition, and the objections, which we are now about to indicate, will, if our view of the subject be substantially correct, be found to apply to it in its twofold aspect,—that is to say, some of them will apply to it independently as a treatise on Form and Magnitude; some of them will apply to it independently as a treatise on scientifically systematised ratiocination; and others will apply to it in that compound (mixed) character, which is usually regarded as its especial and unique prerogative.

In the first place we would remark that many of the facts or cases belonging to the science of form and magnitude, may be equally well exposited in the form of Theorem or of Problem. In some cases, however, especially those of a very general and comprehensive

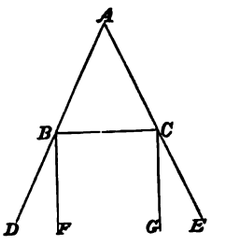
character, the "Theorem" commends itself as the naturally preferable and almost indispensable form of exposition, whilst, in other cases, especially those of a more particular and constructive character, the "Problem" makes a readily admitted claim for preference. Since, on introducing a student to any department of knowledge, it is an object to make the entrance as intellectually agreeable and inviting as the circumstances justify, there may be, we think, good and sufficient reasons why a series of simple *Problems* (exclusively) would constitute the most advantageous commencement and introductory part of an elementary treatise on "Form and Magnitude." Take, for instance, the very excellent example which occupies the first place in Euclid's first book.—Something has to be *done;* a triangle is to be constructed with its three sides equal, and one of those equal sides is to be a given straight line; and, also, the construction must be made in such a way that the constructor can show with mathematical certainty that it has been done correctly. Now, although no entirely inexperienced student would know, or even be able to find out without much trouble, how to fulfil the conditions, yet, when shown, the process is easily followed, and the reasoning by which it is justified readily apprehended and appreciated, and this in so complete a manner that the intelligent student distinctly discerns through the construction and demonstration the *rationâle* upon which they are based, and which has originated them. The constructive curiosity and ingenuity of the mind are first excited by the proposal to make something in a particular manner. The mind is then satisfied by the thing being apparently well done; and, lastly, is intellectually gratified by the intelligible reasons for a reliable conviction that it has been done with complete and perfect rectitude. The youthful mind feels that it has acquired some real knowledge of a useful and elevating character, and goes on its way rejoicing. We believe careful enquiry would result in showing that many young students who, having been much pleased with the three first propositions (*Problems*) of Euclid, had almost made up their minds to like the study of geometry very much, found, after *getting through* the three next propositions (*Theorems*) that they had no taste for geometry; and perhaps, after further experience, that they disliked and detested geometry more than any other kind of *learning*.

Let us take, for a second example, the fifth Prop. of Euclid's first book:—*The angles at the base of an isosceles triangle are equal, and, if the equal sides be produced, the angles on the other side of the base shall be equal to one another.* A young student, gifted with a naturally quick judgment (a very distinct gift, be it observed, from that of the faculty of correctly reasoning), will, on reading the statement and looking at the diagram, see at once that it must be as stated. But, on finding the statement followed by a considerable quantity of somewhat intricate and elaborate argument, he may be apt to say to himself, "What can all this be about? Either there must be some deep purpose in it which I do not understand, or else it would seem to be very foolish." Another learner, with a less quick apprehension, looking to the argument itself for explanation of the meaning, might become first perplexed and then mortified and disheartened at finding himself unable to follow an argument, which his teacher would probably tell him was very plain and easy.

In the old nursery question for young children, "If Tom's father is Dick's son, what

relation is Dick to Tom?" the exercise or amusement consists in bothering the child by cautioning and officiously assisting it with repetitions of the question, "Not Tom to Dick, but Dick to Tom. Mind the question. What relation is Dick to Tom; not Tom to Dick," and so on. The answer, we believe, when one is procurable, not wholly irrelevant, usually comes as "Grandson," showing, in such a case, that the child's unpractised brain-mechanism has, under the exciting stimulus of the artificial irritant, excogitated an inverted conclusion. Whether Euclid's fifth Prop. may not have a somewhat similar effect on persons a little older, may be worth considering.

We submit that the following furnishes all the demonstration requisite and desirable :— Let *A B C* be the isosceles triangle. Produce the equal sides *A B* and *A C*, through the points *B* and *C*, to *D* and *E* respectively. From *B*, perp. to *B C*, and from *C*, also perp. to *B C*, draw *B F* and *C G*. Then because angles *F B C* and *G C B* are both right angles they are equal; and because angles *D B F* and *E C G* are each manifestly equal to half of the angle *B A C*, they also are equal. Therefore the whole angle *D B C*, compounded of the angles *D B F* and *F B C*, is equal to the whole angle *E C B*, compounded of the angles *E C G* and *G C B*.

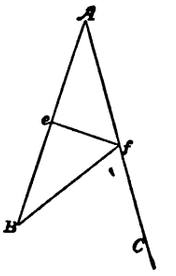


An allied theorem, somewhat less obvious and perhaps more important, is not only omitted from the introductory Props., but is not anywhere directly furnished by Euclid. (It may be perhaps considered as contained in some of the Props. on triangles.) As we may have occasion to refer to it hereafter, we will call it :—

THEOREM A.

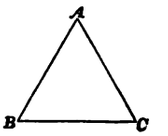
If one of the arms of an acute angle be bisected, and from the point of bisection, at right angles to the arm, a line be drawn to intercept the other arm, and from the point in which the second arm is intercepted a line be drawn to the extremity of the bisected arm, the resulting (bisected) triangle is isosceles and its two opposite angles are equal.

Let *B A C* be any acute angle. Bisect the arm *A B* in point *e*. From *e*, at right angles to *A B*, draw *e f* intercepting the arm *A C* in *f*, and from *f*, to the extremity *B* of the arm *A B*, draw *f B*. Because *e A* and *e B* are equal, and the angles *f e A* and *f e B* are both right angles, and the side *e f* is common to both, the two lesser triangles *f e A* and *f e B* are similar and equal. Therefore the greater triangle *f A B* is isosceles, its two sides *f A* and *f B* are equal, and its opposite angles *f A B* and *f B A* are equal.



If, however, Theorems be admitted amongst the introductory propositions in an elementary treatise on form and magnitude, in which Euclid's first proposition retains its place, it seems to us that the second proposition should be the Theorem :—*All equilateral triangles are equiangular.*

In the equilateral triangle $A\,B\,C$, because the sides $A\,B$ and $A\,C$ are equal to the sides $C\,B$ and $C\,A$, and the base $B\,C$, opposite the vertex point A, equals the base $A\,B$ opposite the vertex point C, the angles $B\,A\,C$ and $A\,C\,B$ are equal. For, since the arms of the one angle are equal to the arms of the other, the base of the one could not equal the base of the other unless the angles were equal. And again, because the sides $B\,A$ and $B\,C$ are equal to the sides $C\,A$ and $C\,B$, and the base $A\,C$ opposite the vertex point B equals the base $A\,B$ opposite the vertex point C, the angle $A\,B\,C$ equals the angle $A\,C\,B$. But angles $A\,C\,B$ and $B\,A\,C$ are equal. Wherefore the three angles are equal, and the equilateral triangle is equiangular.*



Let us now pass on to a more difficult class of propositions in which the student is constantly harassed by being required to refer to other propositions, some of them, perhaps, singly of greater difficulty and complexity than that upon which he is desirous to concentrate his attention exclusively. In some instances he may be inclined to infer, not altogether unreasonably perhaps, that the propositions have been arranged with a view to make references from one to the other indispensably requisite. The important and somewhat difficult Prop. 10. of Book IV. may serve for an example of this. In it we find two references in the Construction, one of them, at the very commencement, to the eleventh of the Second Book, a difficult proposition to which preliminary reference must be made for directions how to proceed with the Construction itself. Then in the Demonstration there are eight references: five of them to propositions in Book I., two to Book III., one to Book IV. It is true that several of these references appear to be practically unnecessary; but there they are, and one of them, again, is to a proposition (the 37th of Book III.) of considerable difficulty, to which careful reference is necessitated, because the demonstration (of Book IV. 10) is made entirely dependent upon it.†

We do not see why, in this instance, a comparatively self-contained proposition, such as the following, might not be preferable, unless, indeed, the variety of subjects brought under mental contemplation at one and the same time, from a sort of bird's eye point of view, in flying about from prop. to prop., may be considered as affording in itself an exhilarating and salutary intellectual exercise.

* In Euclid this theorem is made a corollary to Prop. 5, which, if the student be in any degree perplexed by the elaborate demonstration, is likely to increase his perplexity; for, so far from being really a corollary to the Prop., it is only quite remotely and indirectly related to it.

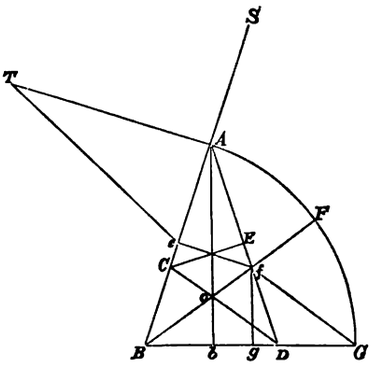
† With regard to the observations which follow, we should not like to be misunderstood as intending to disparage the great merit of Euclid's 2—11, as an isolated problem. The demonstration of the occult relationship between the specific lineal and specific areal magnitudes, which is its subject, is very suggestive, satisfactory, and instructive.

As a whole the proposition evidences much ingenuity, and may be considered as constituting a distinct discovery in the science of "form and magnitude."

PROBLEM B.

To describe an isosceles triangle, having each of the angles at the base double the third angle.

Take any straight line AB. Bisect the line in e, and from point A, perp. to AB, draw AT equal to AB. Join eT. Produce BA through A, and with centre e and distance eT cut the production of BA in point S. With centre A and radius AS cut AB in point C. Then with centre A and radius AB, describe an arc, and with centre B and distance Bd* equal to AC cut that arc in point D. Join AD and BD. The isosceles triangle ABD shall be the required triangle.



Produce BD through D, and with centre B and radius BA describe the arc AFG, meeting the production of BD in point G. From e, perp. to AB, draw ef, intercepting AD in f. Through f draw Gd intercepting AB in d, and from f, perp. to BG, draw fg. Then because ef is perp. to AB, of which e is the point of bisection, Af equals fB, and angle fBA equals angle BAf (Theor. A.). Join Cf.* Then, because Cf and BD are parallel, angle AfC equals angle ADB. But BD equals AC and fB equals Af, and by Prop. 12, *If in two triangles, one of the angles in one of them be equal to one of the angles in the other, and two of the sides in the one be equal to two of the sides in the other, the two triangles are similar and equal,* therefore triangle ACf equals triangle BDf, and Cf equals fD. But BG equals AB, and, since BD equals AC, DG equals CB, and therefore the triangles DfG and CfB are similar and equal. Consequently the greater triangles AfB and BfG are also similar and equal. Hence angle GBf equals angle ABf, which has been shown to equal angle BAf. Wherefore the whole angle ABD, at the base, is double the vertical angle BAD of the isosceles triangle so constructed. *Q. E. F.*

In the foregoing—the directions for the Construction are given, and no reference elsewhere is necessitated. Moreover it might be claimed perhaps that the demonstration is altogether rendered more simple and readily intelligible. However that may be, this Construction as well as that of the original proposition (Euclid's 4-10) is really based on Euclid's 2-11, and the serious defect which we now wish to point out with regard to many of the propositions in Euclid, is preeminently exemplified in that one : namely, so far from making manifest the rationâle or simple basis by a knowledge of which the complex fact becomes intelligible, Euclid is satisfied to demonstrate the compound or complex fact as such, and to leave the student mystified rather than enlightened. The student able to follow the demonstration of the proposition (*Euclid's* 2-11) is satisfied as to its soundness, and apprehends that the fact is indisputable. But, when he has got that far, does he understand the *why* and *wherefore* any more than at

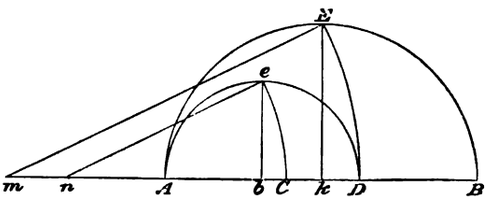
* The line Cf and the point d (in the line AB) are omitted, in order to avoid complicating the figure.

first? He has learnt that when a line is divided in the manner and by the process prescribed, "the rectangle contained by the whole line and one (*sic*) * of the parts shall be equal to the square on the other part." But, for aught he sees or is taught to the contrary, may not this be a quite exceptional phenomenon, a sort of curious coincidence without any necessary connection with what he has learnt in the intelligible elementary propositions preceding it? The two lines are not shewn to have any specific lineal relationship, except that, by making (on the one hand) one of them the side of a square and (on the other) compounding the two into a rectangle, a specific areal relationship (equality) between the two dissimilar figures, so produced, is demonstrable. If then it be the case that he is left in the dark with regard to this proposition, how is it possible that others which refer to and are dependent on it as their basis, can be intelligible? In the interesting example we have taken for examination, namely—the *lineal* problem of how to construct a triangle having two of its angles each double the third angle, the student is told to go to Book 2—11, and find out how to divide a line so that a certain *areal* relationship shall exist between a square constructed upon one part, and a rectangle compounded by the other part and the line itself. We will endeavour to illustrate and make more specific the allegation that something herein is defective by showing the kind of amendment which seems to be needful, and we will do this by taking again the same problem, and treating it, with regard to the Construction, in a manner essentially different.

Problem C.

To divide a given straight line so that the greater shall have to the lesser part the same ratio which the whole line has to the greater part. †

Let $A\,B$ be the given line. Bisect $A\,B$ in k, and with centre k and radius $k\,A$ describe the semi-circle $A\,E\,B$. Produce $A\,B$ through A, and in the produced line take $m\,A$ equal to $k\,A$. From k perp. to $A\,B$, draw $k\,E$ intercepting the semi-circle in point E. Join $m\,E$, and with centre m describe the arc $E\,D$ intercepting $A\,B$ in point D. The given line $A\,B$ shall be divided as required in the point D.



Bisect $A\,D$ in point b, and, in the produced line $A\,m$, take $A\,n$ equal to $A\,b$. From b draw a line perp. to $A\,B$, and from n draw a line parallel to $m\,E$ meeting the perpendicular in point e. With centre n describe the arc $e\,C$.

* It should read: By the whole line and the lesser of the parts. Of the whole line and the greater of the parts the statement would be manifestly untrue.

† Euclid designates this by a technical phrase, and calls it "cutting a line in extreme and mean ratio." But he seems to have supposed that the only way of making the required division was by that complex and somewhat *queer* method which we have already sufficiently dwelt upon. (See *Euclid* 6—30.)

Then, as $Ab : Ak :: AC : AD :: AD : AB$. Since Ak is half of AB, and Ab half of AD, bk equals half of DB.* Then, again, as $Ab : bk :: AC : CD :: AD : DB :: DB : CD$. That is; as AD, the greater, is to AC, the lesser, so is DB, the greater *diff.* (or remainder), to CD, the lesser *diff.* (or remainder). Consequently AC and DB both have the same ratio to CD, and AC equals DB. But $AC : AD :: AD : AB$. Wherefore $DB : AD :: AD : AB$.

Problem D.

(See the fig. of Prob. B.)

To describe an isosceles triangle having each of the angles at the base double the third angle.

Take any straight line AB and divide it in the point C so that the greater part AC has the same ratio to the lesser part CB, as the whole line AB has to the greater part AC (Problem C). Then, with centre A and radius AB describe an arc, and with centre B and distance Bd—equal to AC, cut that arc in point D; and so on, proceeding with the completion of the Construction as already directed in Problem B. The demonstration of Problem B it is needless to repeat. If preferred, however, the completion of the Construction and Demonstration can be taken from Euclid's IV. 10. See also the fig. and the demonstration to the "Tentative Problem" P (Decagon), page 78.

Tentative Problems.

To criticise in any degree unfavourably Euclid's Elements—a work to which modern mathematics owes so much, must appear, at best, a somewhat ungracious proceeding; and now, in the hope of making some amends, we will submit for the mathematical reader's consideration a novelty—which we propose to call "Geometrical Demonstration, with a Tentative Construction." The suggested process, in those exceptional cases to which only it is proposed to apply it, is this:—the Construction may be first made *approximately*—that is, in a manner more or less incorrect, or not certainly known to be correct. But then, as a first step in the demonstration, or, perhaps it would be better to say, preliminary to the demonstration, the Construction must be geometrically corrected; and, finally, its mathematical rectitude proved by strict formal demonstration in the usual way. Its purpose and mode of application will be best shown by some examples, after which we will offer a few remarks in favour of its *limited* admission for certain purposes.

Problem M. (Heptagon.)

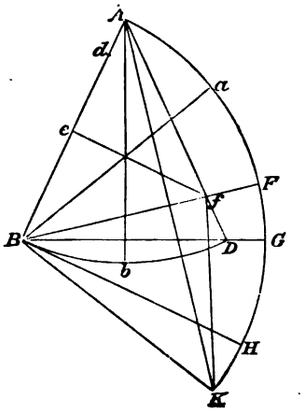
To describe an isosceles triangle having each of the angles at the base equal to one and a-quarter of the vertical angle.

Take any straight line AB and divide it into seven equal parts. With centre A and radius AB describe an arc, and with centre B and distance containing six† of the equal

* Hence $Ak = BD + kD$ = twice $bk + kD$. And Ab = half of $AD = bk + kD$. It follows that $bk : kD :: kD : bC$, and therefore $bk = bC + kD$ and $Ck = kD$.

† The actual *correct* proportion, instead of six to seven, is approximately 6·872 to 1.0

divisions of the line BA, cut that arc in point D. Join AD and BD. The triangle ABD is the *Tentative Construction.*



Bisect line AB in point e, and from e, perp. to AB, draw ef meeting line AD in point f. Bisect angle BAD in line Ab, and bisect the half-arc bD in point c.* Through c from point A draw a line, and with centre B and radius BA describe arc $AFGK$, meeting the line so drawn in point K. Join fK, and produce lines Bf and BD to points F and G in arc $AFGK$. From B, perp. to AB, draw BH to point H in the same arc. Try whether the lines fA and fK are equal, and, if not, move the line BG a little nearer to or further from point F, in arc $AFGK$, as may be requisite, until (the lines AD, Af, fK, AK, Ab, being adjusted accordingly†) the lines Af and fK are equal. The triangle ABD, when so rectified, is geometrically constructed and shall be the required isosceles triangle.

Because Af and fK are equal, and also BA and BK are equal, line AK is at right angles to line Bf (by which it is bisected). And because AK is perpendicular to BF and Ab perp. to BG, and also because AK, drawn through c, bisects arc bD, angle KAb equals angle KAD, and angles KAD and FBG are equal. Therefore angle FBG equals half of the angle bAD, which is half of the whole vertical angle BAD. But, because e is the point of bisection of AB and ef is perp. to AB, Af and Bf are equal, and angle ABF equals angle BAf. Wherefore the entire angle ABG, at the base of the isosceles triangle, exceeds the vertical angle BAD by one-fourth of the angle BAD.—*Q. E. F.*

(*Scholium.*—Since the two angles at the base together equal two and a-half times the vertical angle, the semicircle, of which each of the equal sides of the triangle is a radius, contains three and a-half of the vertical angle, and, accordingly, the entire circle must contain seven such angles).

Problem N. (Octagon.)

To describe an isosceles triangle having each of the angles at the base equal to one and a-half of the vertical angle.

Divide a given straight line AB into eight equal parts. With centre A and radius AB describe an arc, and with centre B and distance containing six‡ of the equal parts of BA cut that arc in point D. Join AD and BD. The isosceles triangle ABD is the *Tentative Construction.*§ Bisect line AB in point e and draw, perp. to AB, line ef, meeting line AD in

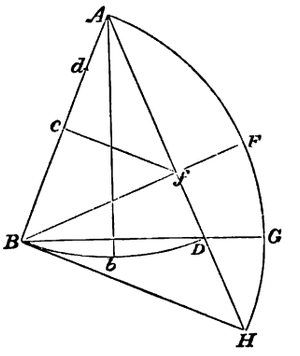
* The engraver has omitted the letter c; and the line AK is a little too near the point D.

† The point D must always be the point in which line BG cuts the arc described with centre A and radius AB, and line ef must always be at right angles to line AB.

‡ The correct proportion, instead of six to eight, is approximately 7·652 to 10.

§ It is almost superfluous to observe that there is in this case no necessity for a tentative construction. It is merely given here, uniformly with the others, as an illustration, in order to make more readily clear the *rationale* of the method pursued. The required triangle, ABD, may be at

point *f*. Produce *A D*, through *D*, and with centre *B* and radius *B A* describe arc *A F H*, meeting the production of *A f* in point *H*. From *B*, through *f*, draw *B F* to meet the arc in point *F*, and produce *B D* through *D* to meet the arc in point G. Try whether the lines *A f* and *f H* are equal, and if not, move the line *B G* a little nearer to or further from the point *F*, in the arc *A F H*, as may be required, until (the lines *A D H* and *B F* being also adjusted accordingly) *A f* and *f H* are equal. The triangle *A B D* so rectified is then geometrically constructed, and shall be the required isosceles triangle. Because point *f* bisects line *A H*, line *B f* is perp. to *A H*. Bisect angle *B A D* in the line *A b*. Then, because *B G* is perp. to *A b*, angle *F B G* equals angle *B A b*; that is, equals half of angle *B A D*; and, since point *e* bisects *A B*, and *e f* is perp. to *A B*, angle *A B F* equals angle *B A D*. Wherefore the whole angle *A B G* at the base exceeds the vertical angle *B A D* by one-half of angle *B A D*.—*Q. E. F.*

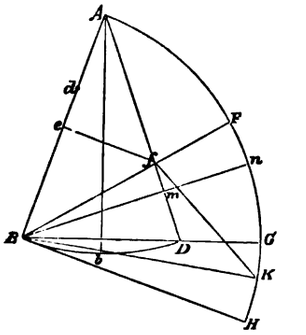


Problem O.—(Nonagon).

To describe an isosceles triangle having each of the angles at the base equal to one and three-quarters of the vertical angle.

Divide the given straight line *A B* into nine equal parts. With centre *A* and radius *A B* describe an arc, and with centre *B* and distance containing six* of the equal parts of *A B* cut that arc in point *D*. Join *A D* and *B D*. The isosceles triangle *A B D* is the *Tentative Construction*.

From point *B*, perpendicular to *A B* draw *B H*, and, with centre *B* and radius *B A*, describe arc *A F H* meeting *B H* in *H*. Bisect *A B* in *e*, and, perpendicular to *A B*, draw *e f* meeting *A D* in *f*; and from *B* through *f* draw *B F* meeting arc *A F H* in *F*. Produce *B D* to meet arc *A F H* in *G*. Bisect arc *G H* in *K*, and join *f K*. Try whether *f A* and *f K* are equal, and, if not, move the line *B G* a little further from or nearer to point *F*, in arc *A F H*, as may be requisite, until (the lines *A D*, *B K*, *f K*, being also adjusted accordingly†) the lines *A f* and *f K* are equal. The triangle *A B D*, so rectified, is the geometrical construction and shall be the required isosceles triangle.



once geometrically constructed by drawing *A H* perp. to *A B* and describing the arc *A H*, then bisecting *A H* in *F* and *F H* in *G*, and drawing the line *A H* cutting line *B G* in *D*. Also (of course) a regular octagon may be directly inscribed in a circle by simple bisection of the quadrants.

* The correct (actual) proportion, instead of six to nine, is approximately 6·84 to 10.

† The point *D* must always be that in which line *B G* cuts the arc described with centre *A* and radius *A B*. Also *e f* must always be at right angles to *A B*.

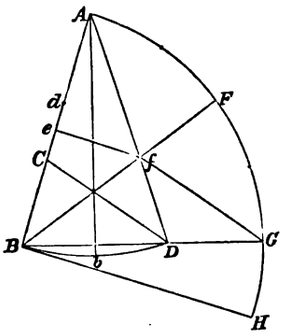
Bisect angle $B\,A\,D$ in line $A\,b$. Then because $B\,H$ is perpendicular to $A\,B$ and $B\,G$ perpendicular to $A\,b$, angle $G\,B\,H$ equals angle $B\,A\,b$, and angle $G\,B\,K$ is half of angle $G\,B\,H$. Therefore angle $G\,B\,K$ equals one-fourth of angle $B\,A\,D$. But, because e bisects and $e\,f$ is perpendicular to $A\,B$, angle $A\,B\,F$ equals angle $B\,A\,D$; and because $f\,K$ equals $f\,A$, angle $F\,B\,K$ equals angle $A\,B\,F$. Wherefore angle $G\,B\,A$, at the base of the isosceles triangle is one-fourth less than twice the angle $B\,A\,D$, and consequently exceeds the vertical angle $B\,A\,D$ by three-fourths thereof.—*Q. E. F.*

(*Scholium.*—Since the two angles at the base are together equal to three and a half of the vertical angle, a semicircle, of which each of the equal sides of the isosceles triangle is a radius, will contain four and a-half, and the entire circle nine, such angles. The vertical angle, accordingly, equals 40 degrees).

Problem P. (Decagon.)

To describe an isosceles triangle having each of the angles at the base equal to twice the vertical angle.

Divide the given straight line $A\,B$ into ten equal parts. With centre A and radius $A\,B$ describe an arc, and with centre B and distance containing six* of the equal parts of $A\,B$, cut that arc in point D. Join $A\,D$ and $B\,D$. The triangle $A\,B\,D$ is the *Tentative Construction.* Bisect $A\,B$ in e, and, perpendicular to $A\,B$, draw $e\,f$, meeting $A\,D$ in f. From B, at right angles to $A\,B$, draw $B\,H$, and, with centre B and radius $B\,A$, describe arc $A\,F\,H$, meeting $B\,H$ in H. Through f produce $B\,f$, and through D produce $B\,D$ to meet the arc $A\,F\,H$ in the respective points F and G. Join $f\,G$, and try whether $f\,A$ and $f\,G$ are equal. If not, move line $B\,G$ a little nearer to or further from the point F, as may be requisite, until (the lines $A\,D$, $f\,G$, $B\,F$, being also adjusted accordingly†) the lines $f\,A$ and $f\,G$ are equal. The triangle $A\,B\,D$, so rectified, is the geometrical construction, and shall be the required isosceles triangle.



Because $A\,f$ and $f\,G$ are equal, the angles $f\,B\,A$ and $f\,B\,G$ are equal. And because e bisects and $e\,f$ is perpendicular to $A\,B$, angle $f\,B\,A$ equals angle $B\,A\,f$. Wherefore the whole angle $A\,B\,G$, at the base of the triangle, is equal to twice the vertical angle $B\,A\,D$.—*Q. E. F.*

Corollary.—Bisect angle $B\,A\,D$ in line $A\,b$. Then, because line $H\,B$ is perpendicular to $A\,B$ and $G\,B$ perp. to $A\,b$, angle $G\,B\,H$ equals the half-vertical angle $B\,A\,b$. Therefore the half-vertical angle equals one-fifth of a right angle, and the whole vertical angle $B\,A\,D$ equals one-fifth of two right angles. Consequently, when one of the equal sides of the triangle is made the radius of a circle, that circle contains ten equal angles, each equal to the vertical angle $B\,A\,D$ of the triangle. (The vertical angle accordingly equals 36 degrees.)

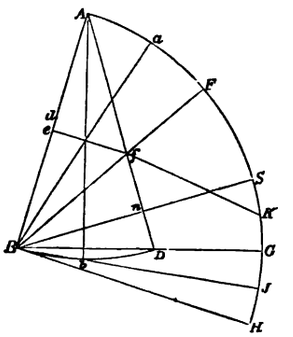
* The correct (actual) proportion, instead of six to ten, is approximately 6·18 to 10.

† See foot-note to the next problem.

PROBLEM Q. (HENDECAGON.)

To describe an isosceles triangle having each of the angles at the base equal to two and a quarter of the vertical angle.

Divide the given straight line into nine equal parts. With centre A and radius AB describe an arc, and with centre B and distance containing five * of the equal parts of BA cut that arc in the point D. Join AD and BD. The triangle ABD is the *Tentative Construction.* Bisect AB in point e, and perpendicular to AB, draw ef intersecting AD in f. From B, at right angles to AB, draw BH, and with centre B and radius BA describe arc AFH meeting BH in H. Produce BD through D, and through f, produce Bf, to meet the arc AFH in the respective points G and F. Bisect arc GH in point J, and in arc AG, on the other side of point G, take GK equal to GJ. Join fK, and try whether the lines fA and fK are equal. If not, move line BG a little nearer to or further from the point H, until (lines AD, fK, BF, being also adjusted accordingly †) the lines fA and fK are equal. The triangle ABD so rectified is the geometrical construction, and shall be the required triangle.



Bisect angle BAD in line Ab. Then because line HB is perpendicular to AB and GB perpendicular to Ab, angle HBG is equal to the half-vertical angle BAb. And because GK equals GJ (*Construction*), angle KBG also equals half of the half-vertical angle BAb, that is, equals one fourth of the whole vertical angle BAD. But because ef is perpendicular to AB, of which e is the point of bisection, angle FBA equals angle BAD, and, since angles FBA and FBK are equal, angle ABK equals twice angle BAD. Wherefore the whole angle ABG, at the base of the triangle, equals two and a quarter of the vertical angle BAD.—*Q. E. F.*

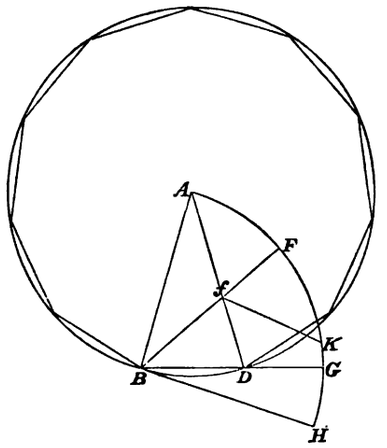
* The correct (actual) proportion, instead of five to nine, is approximately 5·632 to 10.

†. The point D must be always that in which line BG cuts the arc described with centre A and radius AB. Also the line ef must be always at right angles to AB.

Problem R.

*To inscribe an equilateral and equiangular hendecagon in a given circle.**

Take a radius of the given circle for the given straight line ($A\,B$) of Problem Q (with the point A at the centre of the circle), and, as therein directed, construct geometrically an isosceles triangle ($A\,B\,D$) having each of the angles at the base equal to two and a quarter of the vertical angle. Place ten other lines, each equal to the base ($B\,D$) of the triangle round the circumference of the circle. The eleven-sided polygon so inscribed shall be the required hendecagon.



Because the two angles at the base of the triangle are together equal to four and a half times the vertical angle, five and a half of such angles are contained in two right angles, and as the two equal sides of the triangle are radii of the given circle, eleven of the angles must be contained in it. Now each of the other ten lines is equal to that one which is the base of the triangle. Wherefore the eleven sided polygon is equilateral and equiangular, and is inscribed in the given circle.—*Q. E. F.*

(The vertical angle of the isosceles triangle equals, accordingly, 32°·7272 . . .)

The foregoing will suffice as examples of problems with "Tentative Constructions." They are demonstrated geometrically, and, assuming that the demonstrations are not in themselves technically unsound, mathematical certainty is established with regard to the *adjusted* triangles having their angles correlated as required. Why then should not this method be adopted when the geometrician finds himself unable at the outset to give precise directions for the mathematical construction? For example, the draughtsman or other artist asks to be instructed how to inscribe in a circle a regular polygon of seven or of nine sides with geometrical certainty and correctness. Problem M, or O, furnishes him with the triangle, of which the base is one of the sides of the required regular polygon, and all he has to do is to make the centre of the circle the vertex point of the vertical angle, and then to place round the circumference chords each equal to that one which is the base of the isosceles triangle. Why may not he be told to do it so, on the authority of pure geometry?

* This mode of inscribing the equilateral polygon may, of course, be applied to either of the preceding problems. It is one example to serve for the whole.

A superficial objection likely to suggest itself at the first moment, but which will not we think, bear careful consideration, may be thus stated. It may be said: Well, if this were to be allowed, pure geometry would degenerate at once into practical geometry. Any geometrical difficulty might be circumvented and got over in that way. For instance, let it be required to trisect an angle. There are many ways in which the arc subtending the angle may be *approximately* divided into three parts *not far from* equal. Having commenced in that way, then correct geometrically, or *adjust*, the three curvilineal magnitudes so as to make them mathematically equal, and draw the radii to the two divisional points. It would, of course, follow indisputably that the three parts being equal the angle was trisected. Now this, indeed, might be all very well if it could be done ; but to adjust geometrically the three curvilineal magnitudes or their chords is precisely the difficulty in the problem of trisecting the angle. In each of the examples we have given above, there are only two lines to be equalised, and it is evident that the Construction can be geometrically adjusted. Our purpose, however, is merely to submit the case for consideration, not to insist upon it. We are not desirous for the relaxation of any really necessary rule, and, if it can be shown that any element of uncertainty would be, or might be, introduced by the recognition of such a method, as belonging to pure geometry, we should be decidedly against its admission

As to the practical value of the problems themselves, we are about to demonstrate in the next section (Part Second) two distinct and independent methods of trisecting and poly-secting the angle, *i.e.*, of geometrically dividing a given angle into three or into any other required number of equal parts. We shall show that, by the application of either of the two methods, the above problems may be geometrically solved, in the strictest mathematical sense *ab initio*, without the necessity of any "Tentative Construction ;" and also that regular polygons, of any required number, even or odd, of equal sides, may be *directly* inscribed in the circle with geometrical certainty. The intellectual utility and interest of a geometrical problem does not, however, consist solely in its practical application when solved, and we think that the above, although no longer *practically* called for, may still be found worth careful consideration.

In conclusion: To summarize and justify the foregoing criticism of "Euclid's Elements," we would remark that every departmental science is dependent upon its philosophy; upon that philosophy which is not and cannot be satisfied with the mere contemplation of facts or the bare assurance that alleged and apparent facts are really such. The business of philosophy is to reason from the compound and complex to the elementary or more simple constituents of compounded knowledge ; and, conversely, by comparison of particular facts and their correlations, to generalize and combine ; and so, having first secured an assured and safe basis, become able to go forward from a backward to an advanced standpoint, and to ascend by successive stages from a lower to a higher elevation.

That this may be done safely, it is of the utmost importance to ascertain and make sure of the truth and certainty of the supposed facts, for, if an erroneous assumption be mistaken for a fact, the philosophy is itself deceived and then misleads those who trust to it for guidance. Nevertheless, the mere knowledge of even certified facts can be but of little value unless it be vitalized by reasonable philosophy.

Even the science of reasoning itself, considered as a departmental science, may be said to depend upon the correctness of the philosophy applied to it; and, certainly, the science of "Form and Magnitude" is just as much dependent upon *its* applied philosophy as any of the other departmental sciences, such as Chemistry, Astronomy, Geology, &c. What, for instance, would Chemistry be without philosophy, but a mass of lifeless knowledge; a collection of dry inert facts; or, at best, of traditional recipes—a legacy of the philosophy which once vitalized the knowledge?

As wholesome knowledge is the food which nourishes, so are sound reasoning and well-founded belief the vital principle and life-blood of the Intellectual Body; and upon the purity of the Philosophy which animates it, must the well-being, progress, health, and vitality of each Science be dependent—"The spirit of the Living Creature was in the wheels."

Plate 2 illustrates some methods of dividing a given straight line into a required number of equal parts, referred to at the end of Prop. IV.

Plate 2.

Fig. 1

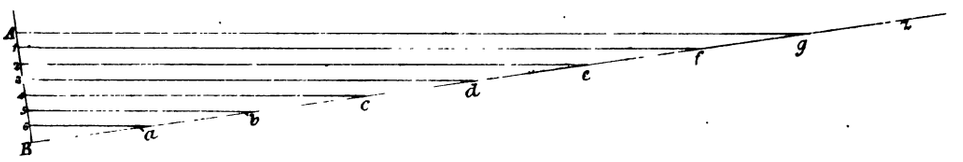


Fig. 4

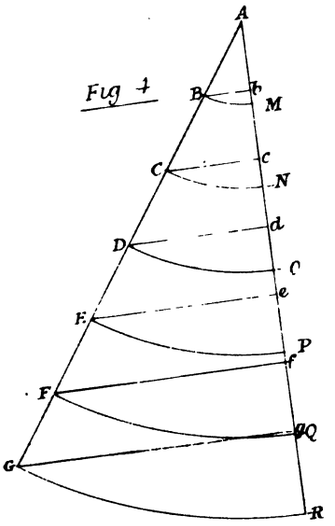


Fig. 5

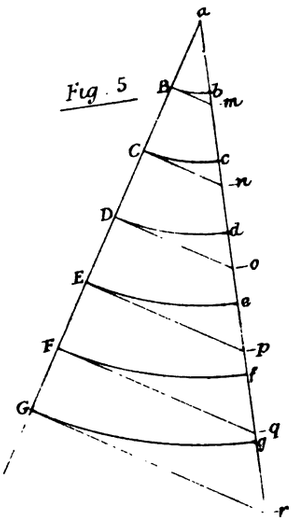


Fig. 2

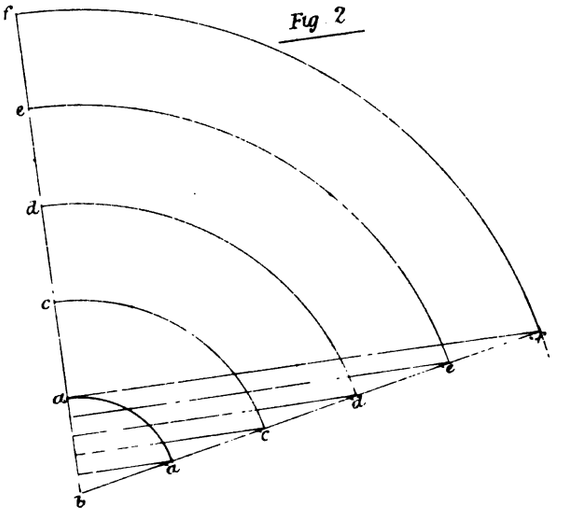
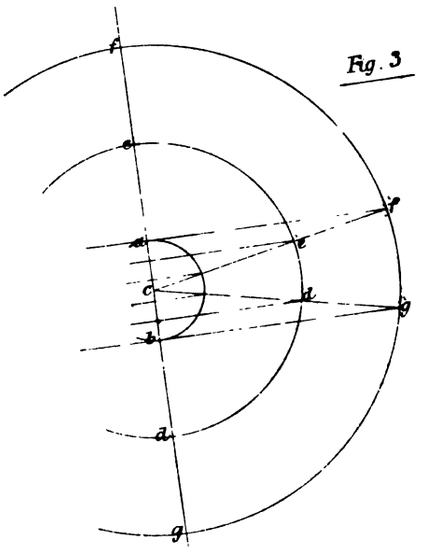


Fig. 3



PART SECOND.

TRISECTION AND POLYSECTION OF A GIVEN ANGLE,

INCLUDING TWO DISTINCT METHODS.

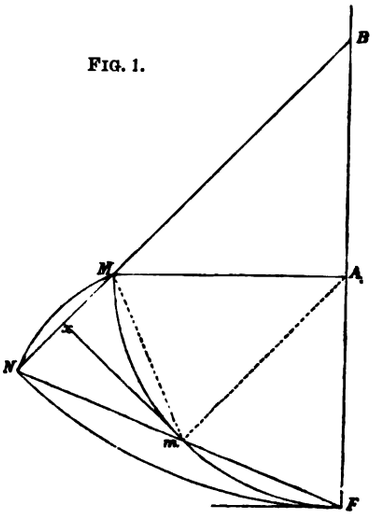
FIRST METHOD OF TRISECTION.

PROP. I. THEOREM.

A.—(1.) If, with the one arm of an angle as a radius, the arc subtending the angle be described, and that original radius be then produced through the vertex of the angle and be duplicated in length, and with the duplicated radius a second arc be described having the same original point (def. 16) as the first arc, and terminated by a radius drawn through the terminal point of the first, the second arc represents the half of the first arc radially duplicated in magnitude; and therefore the chord of the second arc bisects the first arc, and includes the chord of the half of the first arc.

Fig. 1.—Let $M A F$ be a given angle and $F M$ the arc subtending the angle, described with radius $A F$. Produce the arm $A F$ through the vertex A, making $B F$ equal to twice $A F$, and with centre B and radius $B F$ describe the second arc $F N$ terminated by the radius $B N$ drawn through the terminal point M of the first arc. The chord $F N$ of the second arc bisects the first arc in the point m; the half of the chord $F N$ is the chord of the half-arc $F m$, and the second arc $F N$ represents the half-arc $F m$ radially duplicated in magnitude. Join $A m$.

FIG. 1.



(2.) If with the chord of the outer half of the first primary arc as a radius, and the point of bisection of that arc as a centre, a transverse arc be described connecting the terminal point of the first with the terminal point of the second primary arc, that transverse arc will evidently be a similar arc (def. 15) to the second arc, and therefore also similar to the half of the first arc.

Join $m M$ the chord of the outer half of the arc $F M$; and with centre m and

radius $m M$, describe the transverse arc $M N$ connecting the terminal extremities of the arcs $F M$ and $F N$. Bisect the transverse arc $M N$ in the point x.

Because the chord $N M$ of the transverse arc $N M$ is parallel to the primary radius $A m$, the radius $m x$ of the arc $N M$, which bisects the chord $N M$ thereof, is perpendicular to the radius $A m$. Now the radius $m N$ (of the transverse arc) is a part of the primary chord $F N$, of which the other part $m F$ is the chord of the half-arc $m F$. Therefore the radius $m N$, of the transverse arc, is perpendicular to that radius of the first primary arc which bisects the angle $F A m$; and, therefore, the half of the transverse angle $N m M$ equals half of the angle $F A m$, and the whole transverse angle $N m M$ equals each of the angles $F A m$ and $m A M$. Consequently the transverse arc $N M$ and the angle $N m M$ subtended by that arc, are similar to the first primary half-arc $m M$ and the angle $m A M$ subtended by that arc, and also similar to the second primary arc $F N$ and the angle $F B N$ subtended by that arc.

B.—(1.) If the original radius of the second primary arc be duplicated (through its circle's centre), and with that duplicated radius a third primary arc be described, having the same original point (def. 16) as the first and second arcs, and terminated by a radius drawn through the terminal point of the second arc, that third arc represents the half of the second arc radially duplicated in magnitude,* and therefore the chord of the third bisects the second arc, and includes the chord of the half of the second arc.

Fig. 2.—Produce the radius $B F$ through B, making $C F$ equal to twice $B F$, and with centre C and radius $C F$ describe a third arc $F O$ terminated by the radius $C O$ drawn through the terminal point N of the second arc. The chord $F O$ of the third arc bisects the second arc $F N$ in the point n, and $F n$ is the chord of the half-arc $F n$. Join $B n$.

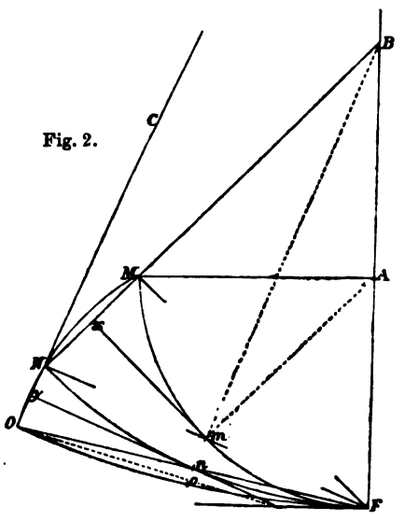


Fig. 2.

(2.) If, with the chord of the outer half of the second primary arc as a radius and the point of bisection of the second primary arc as a centre, a second transverse arc be described connecting the terminal point of the second with the terminal point of the third primary arc, that (second) transverse arc will evidently be a similar arc to the third primary arc, namely, the arc described with the quadrupled radius, and also similar to each of the halves of the (second) arc described with the duplicated radius

* And also represents the fourth part of the first arc, radially quadrupled in magnitude.

Join $n\,N$ the chord of the outer half of the arc $F\,N$, and with centre n and radius $n\,N$ describe the transverse arc $N\,O$, connecting the terminal extremities of the arcs $F\,N$ and $F\,O$. Bisect the transverse arc $N\,O$ in the point y.

Because the chord $O\,N$ of the transverse arc $O\,N$ is parallel to the radius $B\,n$, the radius $n\,y$ which bisects the chord of the arc $O\,N$, is perpendicular to the radius Bn. Now the radius $n\,O$ (of the transverse arc) is a part of the primary chord $F\,O$, of which the other part $n\,F$ is the chord of the half-arc $n\,F$. Therefore the radius $n\,O$, of the transverse arc, is perpendicular to that radius of the second primary arc which bisects the angle $F\,B\,n$; and, therefore, half of the transverse angle $O\,n\,N$ equals half of the angle $F\,B\,n$. Consequently the whole transverse angle $O\,n\,N$ equals each of the angles $F\,B\,n$ and $n\,B\,N$, and also equals the angle $F\,C\,O$ subtended by the third primary arc.

C.—(1.) If the original radius of the third primary arc be duplicated, and with that duplicated radius a fourth arc be described, having the same original point, and terminated by a radius drawn through the terminal point of the third arc, the fourth arc represents the half of the third arc radially duplicated in magnitude, and therefore the chord of the fourth bisects the third arc, and includes the chord of half the third arc.

Fig. 3.

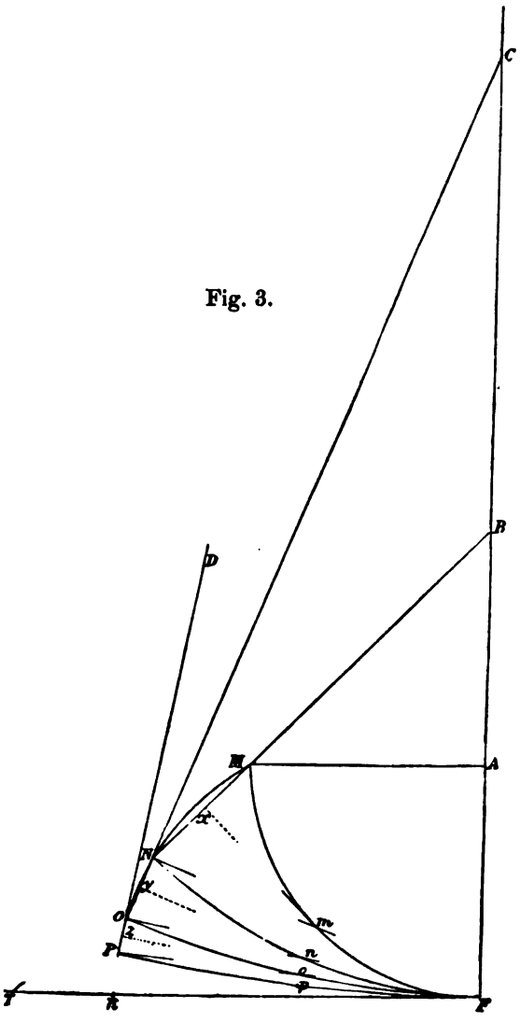


Fig. 3.—Produce the radius $C\,F$ through C, making $D\,F$ equal to twice $C\,F$, and with centre D and radius $D\,F$ describe the arc $F\,P$ terminated by the radius $D\,P$ drawn through the terminal point O of the third arc. The chord $F\,P$ of the fourth arc bisects the third arc $F\,O$ in the point o, and $F\,o$ is the chord of the half-arc $F\,o$. Join $C\,o$.

(2.) If with the chord of the outer half of the third primary arc for a

radius, and the point of bisection of that arc for a centre, a third transverse arc be described connecting the terminal point of the third with the terminal point of the fourth primary arc, that (third) transverse arc will evidently be similar to the fourth primary arc, and also similar to each of the halves of the third primary arc.

Join $o\,O$ the chord of the outer half of the arc $F\,O$; and with centre o and radius $o\,O$ describe the transverse arc $O\,P$, connecting the terminal extremities of the arc $F\,O$ and $F\,P$. Bisect the transverse arc $O\,P$ in the point z.

Because the chord $O\,P$ of the transverse arc is parallel to the radius $C\,o$, the transverse angle $P\,o\,O$ is similar to the angle $F\,D\,P$ subtended by the fourth primary arc, and similar also to each of the angles $F\,C\,o$ and $o\,C\,O$ subtended by the respective halves of the third primary arc.

D.—In like manner the original radius of the fourth primary arc may be duplicated through the circle's centre, and a fifth primary arc be described of sixteen times the radial magnitude of the sixteenth part of the first primary arc; and a fourth transverse arc, similar to the fifth primary arc, be described connecting the terminal extremities of the fourth and fifth primary arcs; and then, again, duplicating the original radius of the fifth, a sixth primary arc* and a fifth transverse arc similar to it, may be described; and so on, in like manner, until the last primary arc be the *ultimate* arc of the circle, namely, one of the least possible divisional magnitudes of cyclal curvature into which the circle can be divided, radially increased to the lineal magnitude of the first primary arc.

Scholium.—Between the ultimate arc and the ultimate chord (which belongs to it) the difference is exceedingly small, and, since increase in radial magnitude is increase in the rectilineal component of the arc and not in its curvature, when the ultimate arc and its chord are increased in radial magnitude to the scale of the circle's octant, the difference between them is less than can be made graphically apparent, and the same line, in a diagram, even when on a large scale, may with virtual correctness represent both of them. The actual difference could be mathematically expressed with precision if we could determine the actual *limit* beyond which the process cannot be carried without the decomposition and destruction of the existent element of lineal curvature then remaining. Having now, however, succeeded in determining the compounded (aggregate) difference in the octant between the sum of its component ultimate arcs and the sum of their ultimate chords, we can readily express, with precision and certainty, the difference between the remaining cyclal curve and its chord, at any given limit to the process of reduction by repeated bisection. But although quite certain and mathematically indisputable, that no cyclal arc (*i.e.*, no part of the circle's circumference) can ever become coincident with its chord, yet the law of approximation belonging to the

* This sixth arc would represent the one thirty-second part of the first primary arc radially increased to thirty-two times the magnitude.

process of repeated bisection can be mathematically carried very far onward towards the actual ultimate *points*, where neither the arc nor its chord *could be* any further divided without destruction of their ideal existence.

We have some time since shown,* by actual trigonometrical measurement of the resulting chord (and correlative lines) at each successive stage of the process, that commencing with the octant and calling the radius thereof unity, the ratio of the chord belonging to the arc which results from the eighth bisection and radial duplication, is numerically expressed by the decimals ·7853969. . . Whilst the ratio of the arc itself (at the same stage) is expressed as ·785398 . . . It is mathematically demonstrable that ultimately, that is—at the vanishing point, the numerical expressions would *apparently* coalesce and become *almost* identical in the quantity ·785397†. . . . a quantity which has been computed to more than two hundred decimal places, and is very well known to mathematicians, but has been hitherto erroneously supposed to express the ratio of the octant's arc to its radius, instead of the ratio of the circle's *ultimate* arc and *ultimate* chord respectively, at the vanishing point, to the radius—when the latter has been increased to the then enormous *radial* scale of the ultimate cyclal fragment, as expressed by the figures. The latter ratio is that which the above figures really and in fact do express. (See Fig. 3, Prop. I., page 85; and See also *Lunar Analysis of the Circle*, Part Third, *page* 119.)

* See Lecture First on the Circle and Straight Line. We have called the process "The Radial Duplication of the bisected Arc." Fig. 3, page 85, may serve for a rough illustration of it.

† The same quantity multiplied by 4, namely, 3·14159 . . . . correctly expresses the ratio of the perimeter of the ultimate (*i.e.*, greatest possible) inscribed regular polygon to the circle's diameter as unity. The Circle's ratio, by Demonstration (Part Fifth), is 3·14269 . . . .

To make the true state of the case as clear as possible: Let us suppose (by means of continued bisection) the computation of the ratios of the inscribed and circumscribed polygons carried to fifty places of decimals, to which extent the ratios of the (*angularly* extremely small) sides of each are expressed by the same figures, and that the succeeding figures are a 5 and a 7 respectively. Now this seems an exceedingly small difference; but, when it is apprehended that the magnitude of the radius has been *re*-doubled at each successive bisection, it will be understood that, in order to make comparison between the complete perimeters of the polygons, the 5 and the 7 respectively have to be multiplied by the enormous multiplier which will then represent the total increase in magnitude of the radius; and so, instead of at the *fifty-first* place of decimals, a very notable difference between the perimeters of the inscribed and circumscribed polygons will be found at the *third* place of decimals. The actual ratio of the *ultimate* circumscribed polygon must be *about* 3·14379 . . .

TRISECTION OF THE ANGLE.

(By the First Method).

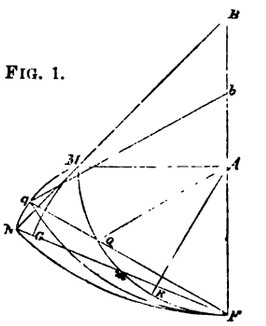
PROP. II. PROBLEM.

To trisect a given angle.

Fig. 1.—Let MAF be the given angle: it is required to divide the angle MAF into three equal angles.

With centre A and radius AF describe the arc FM subtending the given angle: and join FM. Produce the side AF through A, and make BF equal to twice AF. With centre B and radius BF describe the arc FN, terminated by the radius BN drawn through the terminal point M of the arc FM. Draw the chord FN of the second arc bisecting the first arc FM in the point m. With centre m and radius mM describe the transverse arc MN connecting the terminal extremities of the arcs FM and FN. Bisect the radius AF in the point a, and bisect BA in the point b, making the distance bF, equal to three times the distance aF (*i.e.*, equal to one and a half times the radius AF). With centre b and radius bF describe the arc Fq intercepting the transverse arc MN in the point q; and draw the chord Fq of the arc Fq, cutting the first primary arc FM in the point Q. Bisect the arc FQ in the point R, and join AQ and AR. The angle MAF shall be trisected in the radial lines AQ and AR.

FIG. 1.



With centre F and radius FM describe the transverse arc MG intercepting the line FN in the point G.

Evidently the distance GN (that is, the difference between the lines FG and FN) is the quantity of length by which the chord of the radially duplicated half-arc Fm exceeds the chord of the whole primary arc FM; and, since the curvature of the primary arc, continually increasing as its angular magnitude becomes greater, is greatest at its terminal extremity M, it is also evident that, if a part less than the half of the primary arc be cut off from its terminal extremity, and the remaining fractional arc (greater therefore than one-half the primary) be then radially increased to the magnitudinal scale of the whole primary, the excess of length by which the chord of that arc will exceed the chord of the prima, must be less *proportionally to the greater angular magnitude* of the fractional arc, than the excess in the chord of the radially duplicated half-arc. (Preceding Theorem, Prop. 1.) For example: if a fourth part of the primary be cut off, and the remaining three-fourths be then radially increased to the magnitudinal scale of the primary, that increased arc will bisect the

transverse arc $M N$, and the chord of that arc will bisect the transverse arc $M G$; the excess in the chord of the three-fourths arc over the chord of the primary being measured, as before (in the case of the duplicated half-primary), by the part of the chord between the two transverse arcs: and, if one-eighth of the primary arc be cut off, and the remaining seven-eighths thereof be radially increased to the scale of the primary, that increased arc will cut off one-fourth of the transverse arc $M N$, and its chord will cut off one-fourth of the transverse arc $M G$; and so on. Therefore, as the ratio of increase in the chord length of each successive radially-increased fractional arc must be proportional to the diminished curvature in the primary arc, upon which diminished curvature its increase is directly dependent, and as the excess in length of the chords of all radially-increased fractions of the primary greater than the one half and less than the whole primary arc, over the chord of the primary, is contained between the two transverse arcs $G M$ and $N M$, and as the lengthening process commences from the point M (the original extremity of the arc $M N$) and terminates at the point N (the terminal extremity of that arc), and because the arc $N M$ is similar to the primary half-arc $m M$ *(preceding Theorem)* and its curvature the same, corresponding, point to point, with the component incremental parts of the primary half-arc, it follows that the transverse arc $N M$ is the curved line of section which terminates all fractional arcs of the primary—greater than the one-half and less than the whole primary arc, when radially increased to the scale of the primary arc $F M$. Now, if one-third of the primary arc at the terminal extremity thereof, were to be divided off, and it were required to radially increase the remaining two-thirds arc to the magnitudinal scale of the whole primary, it would be accomplished by increasing the radius of the primary by one-half of itself, and with that increased radius describing, from the same original point, an arc similar to the two-thirds arc. Therefore, since radius $b F$ is, by the Construction, equal to three times the half of radius $A F$, and the transverse conn. arc $M N$ must terminate each fractional arc greater in angular magnitude than one half of the primary, when radially increased to the scale of the whole primary, it follows that the arc $F q$, described with centre b and radius $b F$ and terminated by intersection of the transverse arc $M N$ in the point q, represents two-thirds ($F Q$) of the primary arc $F M$, radially enlarged to the scale of the whole of that arc. And because the chord of a lesser similar arc is necessarily included in the chord of the greater arc to which it is similar, when both have the same original point and same tangential line, the chord $F q$, of the greater similar arc $F q$, cuts off two-thirds of the primary arc $F M$ in the point Q, in which it intersects that arc.

Now the point R bisects the arc $F Q$; and $A Q$ and $A R$ are radial lines drawn from the vertex A of the given angle to the points Q and R in the arc $F M$ subtending that angle. Wherefore the two radial lines $A Q$ and $A R$ divide the given angle

MAF into the three equal angles, MAQ, QAR, RAF; and the given angle has been trisected in those two lines, as required.* Q. E. F.

Further; with the same Construction, in Fig. 2:—

Produce the radius BF through the point B and make CF, the produced radius equal to twice BF. With centre C and radius CF describe the arc FO terminated by the radius CO drawn through N the terminal point of the arc FN. Draw the chord of the arc FO bisecting the arc FN in the point n, and with centre n and radius nO describe the transverse arc ON connecting the terminal extremities of the arcs FO and FN. Join nN (the chord of the half-arc nN.) In the radius CF take the point c at three times the distance FA from F (that is, make cF equal to three times the half of BF) and with centre c and radius cF describe the arc Fr intercepting the transverse arc ON in the point r. Draw the chord Fr, of the arc Fr, cutting the second primary arc FN in the point V and the first primary arc FM in the point R.

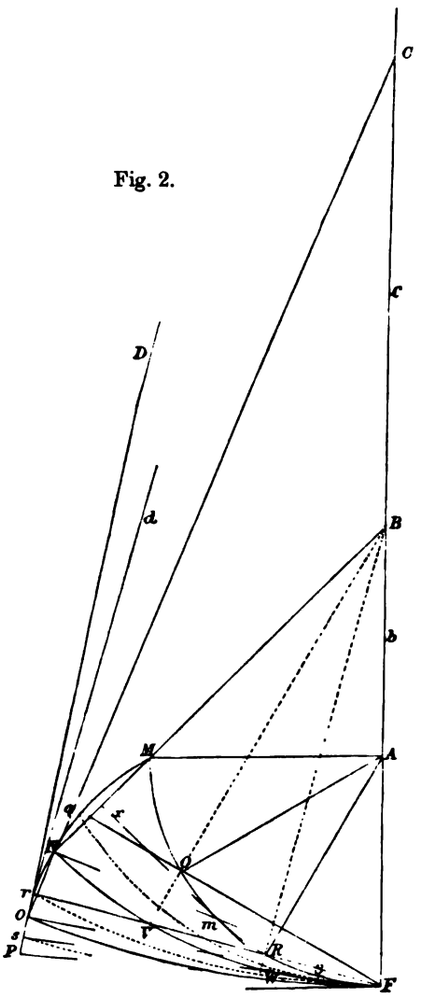


Fig. 2.

With centre F and radius FN describe the transverse arc NH, intercepting the line FO in the point H.

Then, by the reasoning in the preceding demonstration, the transverse conn. arc ON is the curved line terminating all fractions of the second primary arc FN, greater than the one-half and less than the whole arc, when radially increased to the magnitudinal scale of the whole arc; and consequently the arc FV, cut off from FN by the chord of the arc Fr, is two-thirds of the arc FN, because radius cF is proportional to radius BF as three to two. But it also follows, because radius BF is proportional

* We consider this demonstration of the trisection quite complete and conclusive in itself. The purpose of the illustrations and demonstrations which follow will become apparent when they are carefully looked into.

to radius AF as two to one, and because arc FN is similar to the half-arc Fm, and represents that half-arc radially duplicated in magnitude, that the same chord Fr cuts off two-thirds of the half-arc Fm in the point R. Now two-thirds of the half-arc is the same as one-third of the whole arc, and Fm is half of the first primary arc FM. Wherefore the arc FR, cut off from FM by intersection of the chord Fr, is one-third of the arc FM subtending the given angle.

Further, with the same Construction—Fig. 2 :—

Produce the radius CF through the point C and make DF the produced radius equal to twice CF. With centre D and radius DF describe the arc FP terminated by the radius DP drawn through O, the terminal point of the arc FO. Draw the chord of the arc FP, bisecting the arc FO in the point o, and with centre o and radius oP describe the transverse arc PO, connecting the terminal extremities of the arcs FP and FO. Join oO (the chord of the half arc oO). In the radius DF take the point d at six times the distance FA from F (that is, make dF equal to three times BF), and with centre d and radius dF describe the arc Fs, intercepting the transverse arc PO in the point s. Draw the chord Fs, of the arc Fs, cutting the third primary arc FO in the point $\mathfrak{X}$, the second primary arc FN in the point W, and the first primary arc FM in the point S. Then, by the reasoning in the preceding demonstration, the transverse arc PO is the curved line terminating all fractions of the third primary arc FO, greater than the one-half and less than the whole arc, when radially increased to the magnitudinal scale of the whole arc ; and, consequently, the arc Fs, cut off from FO by the chord of the arc Fs, is two-thirds of the arc FO, because radius dF is proportional to radius CF as three to two. And it follows because radius CF is proportional to radius BF as two to one, and arc FO represents the half of arc FN radially duplicated in magnitude, that the same chord Fs cuts off one-third of the arc FN in the point W. And it also follows, because radius CF is proportional to radius AF as four to one, that the same chord Fs cuts off one-sixth of the arc FM* in the point S.

SCHOLIUM.—Evidently, by again duplicating the radius and repeating the process, the one-twelfth of the first primary arc may be cut off by the chord Ft in the point T; and so on.)

COROLLARY.—Because the second primary arc FN represents the half Fm of the first primary arc FM, radially duplicated in magnitude, and because the transverse arc NM is similar to the half-arc Fm or mM (*preceding demonstration*)—the original point N of the one corresponding to the original point F or m of the other, it follows that every component point or component increment in the one arc corresponds to a component point in the other at the same proportional distance from the original and from the terminal points of the arc of which it is a component. For example, a point in the transverse arc NM, at one-fourth the arc-distance of M from the

* That is, cuts off one-third of the half-arc, which is the same as two-thirds of the quarter arc.

original point N of that arc, corresponds to a point in the primary (half)-arc $m\,M$ at one-fourth the arc-distance of M from the original point m of that arc. Now if two-thirds of the primary arc FM be cut off, one-third thereof, equal to two-thirds of the half-arc $m\,M$, must remain; if three-fourths be cut off, one-fourth, equal to one-half the half-arc, must remain; and, generally, if $(n-1)$ nth parts be cut off from the whole primary, one-nth part, equal to two-$n$ths of the half-arc, must remain; and, since the radially increased arc is in each case terminated by the transverse arc of a similar section, the point in which the transverse arc ($N\,M$) is intersected by the radially increased two-thirds arc, must leave two-thirds of that transverse arc remaining, the point of intersection by the three-fourths arc must leave one-half remaining, and, generally, the fractional arc which leaves two-$n$ths of the primary half-arc remaining, must, when radially increased to the scale of its primary, leave two-$n$ths of the transverse arc remaining. Hence, since the chord of the fractional primary arc is included in the chord of the same arc radially increased, it follows that the production of that chord which cuts off three-fourths from the primary bisects the transverse arc; and the production of that chord which cuts off two-thirds from the primary cuts off one-third from the transverse arc in the point in which that produced chord intercepts it.

SCHOLIUM.—Derived from and grounded upon the fact made evident in the foregoing Corollary, another mode of fulfilling the requisition "to trisect a given angle" becomes available; namely, by making the arc subtending the given angle the transverse arc connecting the terminal extremities of a primary arc and the radially duplicated half of that primary. For example :—

PROBLEM.—*It is required to divide a given angle into three equal angles.* (Fig. 4.) Let $M\,m\,N$ be the given angle. Produce the arm $N\,m$ of the angle through the vertex m thereof, and make the produced line $N\,F$ equal to twice $N\,m$. Describe the arc $N\,M$ subtending the given angle. Draw the chord NM thereof and produce it indefinitely through the extremity M of the arc. From the point m, at right angles to the line $N\,F$, draw a line meeting the produced chord of the arc $N\,M$ in the point B. Join $F\,B$. With centre B and radius $B\,F$ describe the arc $F\,N$ (of which the line $F\,N$ is the chord) terminated by the point N at one extremity of the arc $N\,M$. Bisect radius $B\,F$ in the point A; and, with centre A and radius $A\,F$, describe the arc $F\,M$ terminated by the point M at the other extremity of the arc $N\,M$. In the radius $B\,F$ bisect the distance $B\,A$ in the point b; and, with centre b and radius $b\,F$ describe the arc $F\,q$ intercepting the transverse arc $N\,M$ in the point q.

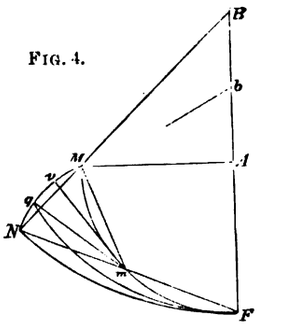


FIG. 4.

Bisect the arc-distance $M q$ (in the arc $N M$) in the point v. Join $m q$ and $m v$. The given angle $N m M$ is trisected in the radial lines $m q$ and $m v$.

Draw the chord of arc $F q$ cutting arc $F M$ in the point Q. Then, because radius $B F$ is to radius $A F$ as two to one, and radius $b F$ to radius $A F$ as three to two, the arc $N M$ subtending the given angle is the transverse arc of similar section belonging to arcs $F M$ and $F N$, and arc $F q$ represents two-thirds of arc $F M$ radially increased to the scale of $F M$. Therefore the chord of arc $F q$ cuts off two-thirds of the primary arc $F M$ in the point Q, and one-third of the transverse arc $N M$ in the point q in which it intercepts the arc $N M$. Now the point v bisects the arc-distance $q M$ (in the arc $N M$). Wherefore the points q and v trisect the arc $N M$ subtending the given angle $N m M$, and the given angle is divided into three equal angles by the radial lines $m q$ and $m v$ as required.

POLYSECTION OF THE ANGLE.

(*By the First Method of Trisection.*)

PROP. III. PROBLEM.

To divide a given Angle into any required number of equal Angles.

(Plate 3, Fig. *a*.) Let $M A F$ be the given angle; and—

1st. Let the required number be four. With centre A and radius $A F$ describe the arc $F M$ subtending the given angle. Produce radius $A F$ through A, and make $B F$ equal to twice $A F$. With centre B and radius $B F$ describe $F N$ terminated by radius $B N$ drawn through the terminal point M of the first arc. Draw the chord $F N$, of the arc $F N$, bisecting the first arc $F M$ in the point K. With centre K and radius $K N$ describe the transverse arc $N M$. In the radius $B F$ take point d at the distance $d F$ greater than $A F$ by one-third of $A F$. With centre d and radius $d F$ describe the arc $F s$ intercepting the transverse arc $N M$ in the point s; and draw the chord $F s$ cutting the arc $M F$ in the point S. Then: because radius $d F$ is proportional to radius $A F$ as four to three, and arc $F s$ is terminated by the transverse arc of similar section $N M$, the arc $F s$ represents three-fourths of the arc $F M$ radially increased to the magnitudinal scale of the whole arc $F M$ (*preceding demonstration*). Therefore, since the chord $F s$ cuts arc $M F$ in point S, the arc $M S$ is one-fourth of arc $M F$. Divide the remaining three-fourths of arc $M F$ into three parts, each equal to $M S$, and to each of the divisional points in the arc draw a radial line from the vertex of the given angle, thereby dividing the given angle into four equal angles as required.

2nd. Let the required number be five (Fig. *b*). In the radius $B F$ take point e at

distance $e\,F$ greater than $A\,F$ by one-fourth of $A\,F$. With centre e and radius $e\,F$ describe arc $F\,t$ intercepting the transverse arc $N\,M$ in point t; and draw the chord $F\,t$ cutting arc $M\,F$ in the point T. Then, because radius $e\,F$ is proportional to radius $A\,F$ as five to four, and arc $F\,t$ is terminated by the transverse arc of similar section $N\,M$ and the chord of arc $F\,t$ cuts arc $M\,F$ in point T, arc $M\,T$ is one-fifth of arc $M\,F$. Divide the remaining four-fifths of arc $M\,F$ into four parts, each equal to $M\,T$, and draw a radial line from the vertex of the given angle to each of the four divisional points in the arc, thereby dividing the given angle into five equal angles as required.

3rd. Let the required number be seven (Fig. *c*). In the radius $B\,F$ take point f at distance $f\,F$ greater than $A\,F$ by one-sixth of $A\,F$. With centre f and radius $f\,F$ describe the arc $F\,u$ intercepting the transverse arc $N\,M$ in point u; and draw the chord $F\,u$ cutting the arc $M\,F$ in the point U. Then, because radius $f\,F$ is proportional to radius $A\,F$ as seven to six, and arc $F\,u$ is terminated by the transverse arc of similar section $N\,M$, and the chord $F\,u$ cuts arc $M\,F$ in point U, arc $M\,U$ is one-seventh of arc $M\,F$. Divide the remaining six-sevenths of arc $M\,F$ into six parts, each equal to $M\,U$, and draw a radial line from the vertex of the given angle to each of the six divisional points in the arc, thereby dividing the given angle into seven equal angles as required.

4th. Let the required number be nine (Fig. *d*). In the radius $B\,F$ take point g, at distance $g\,F$ greater than $A\,F$ by one-eighth of $A\,F$. With centre g and radius $g\,F$ describe the arc $F\,w$ intercepting the transverse arc $N\,M$ in point w; and draw the chord $F\,w$ cutting the arc $M\,F$ in the point W. Then because radius $g\,F$ is proportional to radius $A\,F$ as nine to eight, and arc $F\,w$ is terminated by the transverse arc of similar section $N\,M$, and the chord $F\,w$ cuts arc $M\,F$ in point W, arc $M\,W$ is one-ninth of arc $M\,F$. Divide the remaining eight-ninths of arc $M\,F$ into eight parts each equal to $M\,W$, and draw a radial line from the vertex of the given angle to each of the eight divisional points in the arc, thereby dividing the given angle into nine equal angles as required.

5th. Let the required number be eleven (Fig. *e*). In the radius $B\,F$ take point h at distance $h\,F$ greater than $A\,F$ by one-tenth of $A\,F$. With centre h and radius $h\,F$ describe the arc $F\,x$ intercepting the transverse arc $N\,M$ in point x; and draw the chord $F\,x$ cutting the arc $M\,F$ in the point $\mathfrak{X}$. Then, because radius $h\,F$ is proportional to radius $A\,F$ as eleven to ten, and arc $F\,x$ is terminated by the transverse arc $N\,M$ of similar section, and the chord $F\,x$ cuts arc $M\,F$ in point $\mathfrak{X}$, arc $M\,\mathfrak{X}$ is one-eleventh of arc $M\,F$. Divide the remaining ten-elevenths of arc $M\,F$ into ten parts each equal to $M\,\mathfrak{X}$, and draw a radial line from the vertex of the given angle to each of the ten divisional points in the arc, thereby dividing the given angle into eleven equal angles as required.

Finally. In the like manner, a given angle may be divided into any required number whatsoever of equal angles.

Fig. a.

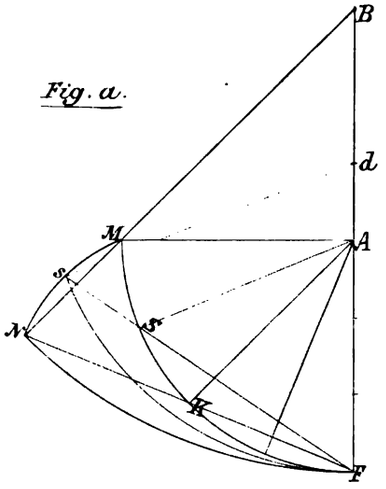


Fig. b.

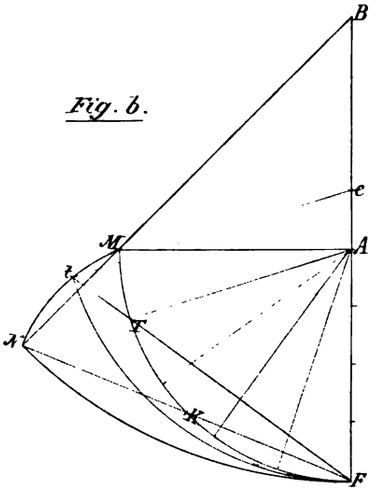


Fig. c.

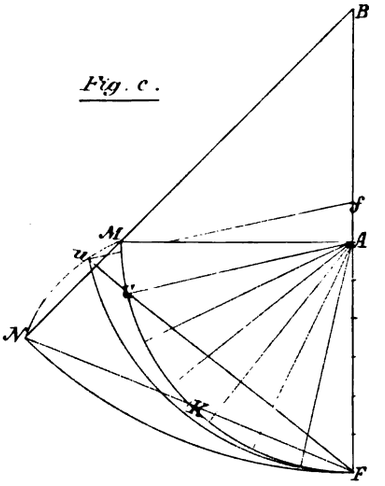


Fig. d.

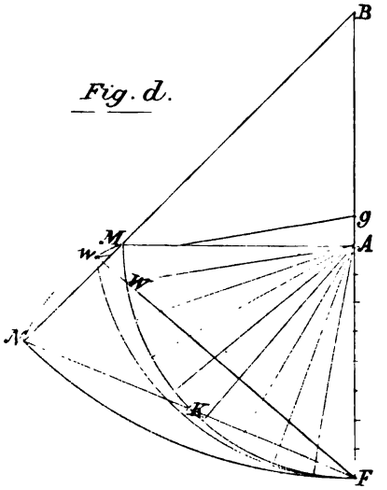
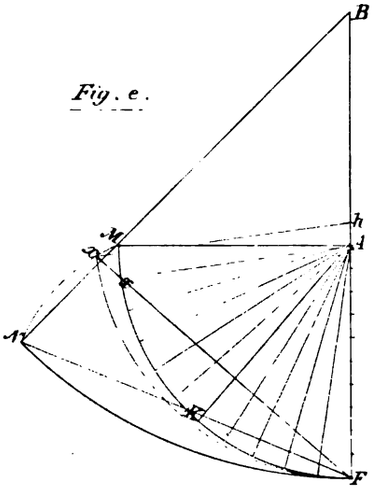


Fig. e.



THE SECOND METHOD OF TRISECTION AND POLYSECTION.

Introductory Note.—Let an arc be bisected, and from the two extremities of the arc draw equal chords to points less distant respectively from each extremity than the point bisecting the arc. An intervening curvilineal space or *connecting-arc* will remain between the interior ends of the two arcs belonging to the chords, and of which intervening connecting arc, if bisected, the point of bisection will evidently be the same as the point bisecting the entire arc. Now let three equal chords be taken, each somewhat less than the chord belonging to a one-third section of the entire arc, and let the three chords be so arranged that the points at one of the extremities of each of two of them coincide respectively with the points at the extremities of the entire arc, and that the middle point of the third chord is coincident with the point which bisects the entire arc. Then let it be required to lengthen the sectional arcs, belonging to the three chords, equally, so that the three arcs, being still equal each to each, shall together constitute the entire arc. This cannot now be done by bisecting each of the two curvilineal spaces or intervening arcs, and adding the halves to the adjacent ends of the equal divisional arcs belonging to the three chords, for, by so doing, the arc belonging to the chord which occupies the middle position would be lengthened twice as much as each of the other two, because half of an intervening arc would be added to each of the ends of that middle divisional arc and only to one of the ends of each of the other two divisional arcs. It, therefore, becomes apparent that the two intervening arcs must be *trisected*,* and then one of the three equal divisional parts of each be added to the one end of the middle divisional arc, and the remaining two-thirds to the end of the adjacent divisional arc, each of which then becomes one of three equal constituents of the entire arc.

The means of rendering the problem solvable by the "Second Method" depends essentially on the discovery of the means of trisecting the intervening connecting-arcs, which, as will be presently explained, consists in the application of two equal chords intersecting each other in the manner shown particularly in the enlarged sections (Figs. 2 & 3, Plate 4) of the Construction belonging to Prop. IV.

* It is practically sufficient to trisect *one* of the curvilineal connecting spaces; because when one of the points of trisection of the entire arc is thus obtained, bisection of the two-thirds part, which remains, will give the three equal components.

Prop. IV. Problem.

To trisect a given angle.

(Plate 4, *Fig.* 1.) Let EAF be the given angle: it is required to divide EAF into three angles equal each to each. With centre A and radius AE describe the arc EKF. Bisect the angle EAF in the radial line AK, and join EF. Find the lateral vertex centre G of the triangle AEF, (Prop. 2, *P. F.*), and through G draw, from the side AE, Be parallel to the side AF, and, from the side AF, Df parallel to the side AE; thus dividing the line EF into three equal parts in the points e and f in which they respectively intercept that line (Prop. 3, *P. F.*). From D, through the point e, at right angles to the chord EF, draw Dq intercepting the arc EKF in the point q. And from B, through the point f, at right angles to the chord EF, draw Bp intercepting the arc EKF in the point p. Join Fp and Eq, cutting and bisecting the lines Dq and Bp respectively in the points g and h. Through the point g draw Ar intercepting the arc EKF in the point r. And through the point h draw As intercepting the arc EKF in the point s. From point s to point r draw the chord sr, and from point F to point q draw the chord Fq, the two chords intersecting each other in the point o. From or cut off one-third in the point n, and through point n draw the radius AN.* With centre N and distance NF cut arc NE in point M, and join AM. The radial lines AM and AN so drawn shall trisect the angle EAF.

Demonstration a.

(1.) Because the point e bisects the distance Ff (in the chord EF), and the lines Bp and Dq are both perpendicular to Ff, and g is the point in which the chord Fp intersects the line Dq, the point g bisects the chord Fp. Now AF, Ar, Ap, are radii of the arc Frp of which Fp is the chord. Therefore the radius Ar, drawn through the point g, bisects the arc Frp, and the arc Fr is equal to the arc pr.

(2.) Because point r bisects arc Frp, and (by the Construction) arc rq equals arc sp, the arc Fq (compounded of Fr and rq) equals the arc rs (compounded of rp and ps); consequently the chord sr equals the chord Fq, and the parts or and oq cut off by mutual intersection of those chords, are also equal. Since qo and or are equal, and the chords of which they are parts are equal (see *the enlarged section of the figure* at Fig. 3) when the radius (A O) is drawn through the point o it is at right angles to the chord qr of arc qr, and bisects that chord. Similarly, when the radius is drawn through point n and intercepts arc EKF in the point N, if with centre N

* See the enlarged sections, Figs. 2 and 3.

and distance $N r$ the arc be cut in point v, opposite to r, the radius $A N$ will be at right angles to the chord $v r$, and bisect that chord; and, also, if $v n$ be joined, and the chords $v N$ and $r N$ be drawn, two similar and equal triangles $v n N$ and $r n N$ will be obtained. Manifestly, therefore, the two arcs $v N$ and $r N$ must be equal. Now if from the line $o q$ the one-third part $o w$ (equal therefore to $o n$ the one-third of $o r$) be cut off, and a radius drawn through the point w, it is obvious that the arc cut off between point q and the point then intercepted by the radius must equal arc $r N$, and therefore also equal arc $v N$. But when the radius ($A O$) passes through the middle point o, the angles $q o O$ and $r o O$ on each side respectively are manifestly equal, and, since $o n$ is one-third of $o r$, and $o w$ one-third of $o q$, the arc cut off from arc $r q$ by the radius drawn through point w must equal twice the arc $r N$, and if $w N$ be joined, the triangles $q w v$, $N w v$, $v n N$, $r n N$, must be similar and equal, for arcs $r N$, $N v$, $v q$, are parts of the same circumference, and the radius is always at right angles to the circumference at the point in which it intercepts that circumference, so that the line $N w$ must equal $r n$ and also equal $q w$. That is to say, the point intercepted by the radius drawn through point w must necessarily be the point v and bisect the arc $q N$. Wherefore the component arc $r N$ is one-third, and the component arc $q N$ two-thirds of the arc $q r$.

(3.) Since points f and e divide the chord $E F$ into three equal parts, arc $E p$ equals arc $F q$; and since s is the point in which radius $A s$, drawn through the point of bisection h of the perpendicular $f p$, intercepts the arc $E K F$, and r is the point in which radius $A r$, drawn through the point of bisection g of the perpendicular $e q$, intercepts the same arc, the sectional arc $s p$ equals the sectional arc $r q$. Hence, taking the equal arcs $s p$ and $r q$ from the equals $E p$ and $F q$, the remainders $E s$ and $F r$ are equal. But, by the Construction, point N bisects arc $F M$ and point r bisects arc $F p$. Hence, arc $p M$ equals twice arc $r N$ (*i.e.*, equals arc $q N$, because $q N$ has been shown to equal twice $r N$), and arc $s M$ equals arc $r N$; and therefore, since arcs $E s$ and $F r$ are equal, and arcs $s M$ and $r N$ also equal, arc $E M$ equals arc $F N$.* Wherefore the three arcs $E M$, $M N$, $N F$, are equal each to each, and the given angle $E A F$ is trisected in the two radial lines $A M$ and $A N$.

Demonstration b (supplementary to the foregoing demonstration).

(Plate 4, *Fig.* 2.) With centre F and radius equal to chord $p q$, cut arc $F q$ in point t.

* If preferred, it may be directly shown, in the same manner as before, by mutual intersection of the chords of arcs $r s$ and $E p$, and cutting off one-third of the sectional arc $s p$ in the point M, that arc $p M$ equals twice arc $s M$, and then, that since arcs $s p$ and $r q$ are equal, arcs $p M$ and $s M$ equal respectively arcs $q N$ and $r N$, and therefore each of the arcs $E M$ and $M N$ equals arc $F N$

From t, through the point g, draw a line of indefinite length, and with centre g and distance $g\,e$ cut that line in the point i. Then, because the chord $p\,q$ equals $f\,e$, one of the three equal divisions of the entire arc's chord, the chord $F\,t$ also equals one of those equal divisions; and since the line $e\,q$, bisected by point g, is perpendicular to the chord $p\,q$, the line $i\,t$, drawn through the same point g, is perpendicular to the chord $F\,t$. Also, because $F\,e$, one of the three equal divisions of the entire arc's chord, and the chord $q\,p$ are equal, the chord $F\,p$ passes through the point g and is at right angles to the radius $A\,r$. And, as chords $q\,p$ and $t\,F$ are equal, the chord of the intervening arc $q\,t$ is also at right angles to the radius $A\,r$ and is bisected by it. Now, by the construction, the middle point of the arc of which $q\,p$ is the chord is coincident with the point of bisection of the entire arc, and the point F, at the extremity of the arc of which $F\,t$ is the chord, is at the extremity of the entire arc. Therefore, when the entire arc is to be divided into three equal components, two-thirds of the intervening arc $t\,q$ must be added to the divisional arc $F\,t$ (*Introductory Note*). It has been shown, in section (2) of the preceding demonstration, that the radius $A\,N$ cuts off one-third ($r\,N$) from the arc $r\,q$, and since the point r has been now shown to bisect the intervening arc $t\,q$, it follows that $q\,N$ is one-third and $t\,N$ two-thirds of that arc, and consequently that the radius $A\,N$ cuts off, in the point N, one-third of the entire arc $F\,K\,E$.*

Demonstration c (supplementary to the foregoing demonstrations).

Repeating in Plate 5, *Fig.* 4, the previous construction (as in *Fig.* 1). Draw the chord $F\,M$ intersecting and bisecting that part $'N\,N$, cut off from radius $A\,'N$ by the entire arc's chord $F\,E$, in the point z. Produce the radius $A\,N$ through N and make the production $N\,'A$ equal to $'N\,A$ (so that $z\,'A$ equals $z\,A$). Join $'A\,M$ and $'A\,F$; and, in point $'D$, cut off from $'A\,M$ the third part thereof (so that, since $'A\,M$ equals $A\,F$, $'A\,'D$ equals $A\,D$). From pont $'D$ perpendicular to the chord $F\,E$, draw $'D\,o\,'C$ cutting the arc $F\,N\,M$ in point o and the radial line $'A\,A$ in point $'x$, and intercepting $F\,E$ in point $'C$. With centre $'A$ and radius $'A\,F$ describe the arc $F\,'N\,M$ cutting the duple radial line $A\,'A$ in point $'N$. Produce the line $'D\,'C$ through $'C$ intercepting the arc $F\,'N\,M$ in point $'q$. Draw the chord $M\,N$ cutting the perpendicular $'q\,'D$ in the point $'e$, and the perpendicular $e\,q$ in the point c. Produce the radius $A\,g\,r$ through r and make $g\,a$ equal to $g\,A$. Join $a\,F$ and $a\,p$. Bisect the distance $'e\,'q$ (in the perpendicular $'D\,'e\,'q$) in point $'g$, and through $'g$ draw the duple radial line $a\,'a$, making $'g\,'a$ equal to $'g\,a$.

* The mode of trisecting arc $q\,r$ as shown in Demonstration *a* (*Fig.* 3) may be also applied to arc $q\,t$ in Demonstration *b*, thus:—(Fig. 2) With centre t and distance equal to $F\,q$ cut arc $r\,E$ in point w; and draw the chords $F\,q$ and $w\,t$ cutting each other in point R. From $R\,q$, the extremity of the chord $F\,q$, cut off one-third in point $'n$, and through $'n$ draw the radius $A\,N$. Now if from $R\,t$, the extremity of $w\,t$, one-third be cut off in point $'m$ and through $'m$ the radius $A\,'N$ be drawn, it will follow that arcs $^{t}\,'N$, $'N\,N$, $N\,q$, are equal, for if $'N\,'n$ be joined that line must manifestly equal $t\,'m$ and also equal $q\,'n$. Therefore the arc $q\,t$ is trisected in the points N and $'N$; and arc $q\,N$, which was shown (in Demonstration *a*) to be two-thirds of arc $q\,r$, is now independently shown to be one-third of arc $q\,t$ of which r is the point of bisection. (Hence it follows that the radius, drawn through point $'n$, also passes through point n.)—See *the Enlarged Sections Fig.* 3.

Plate 5

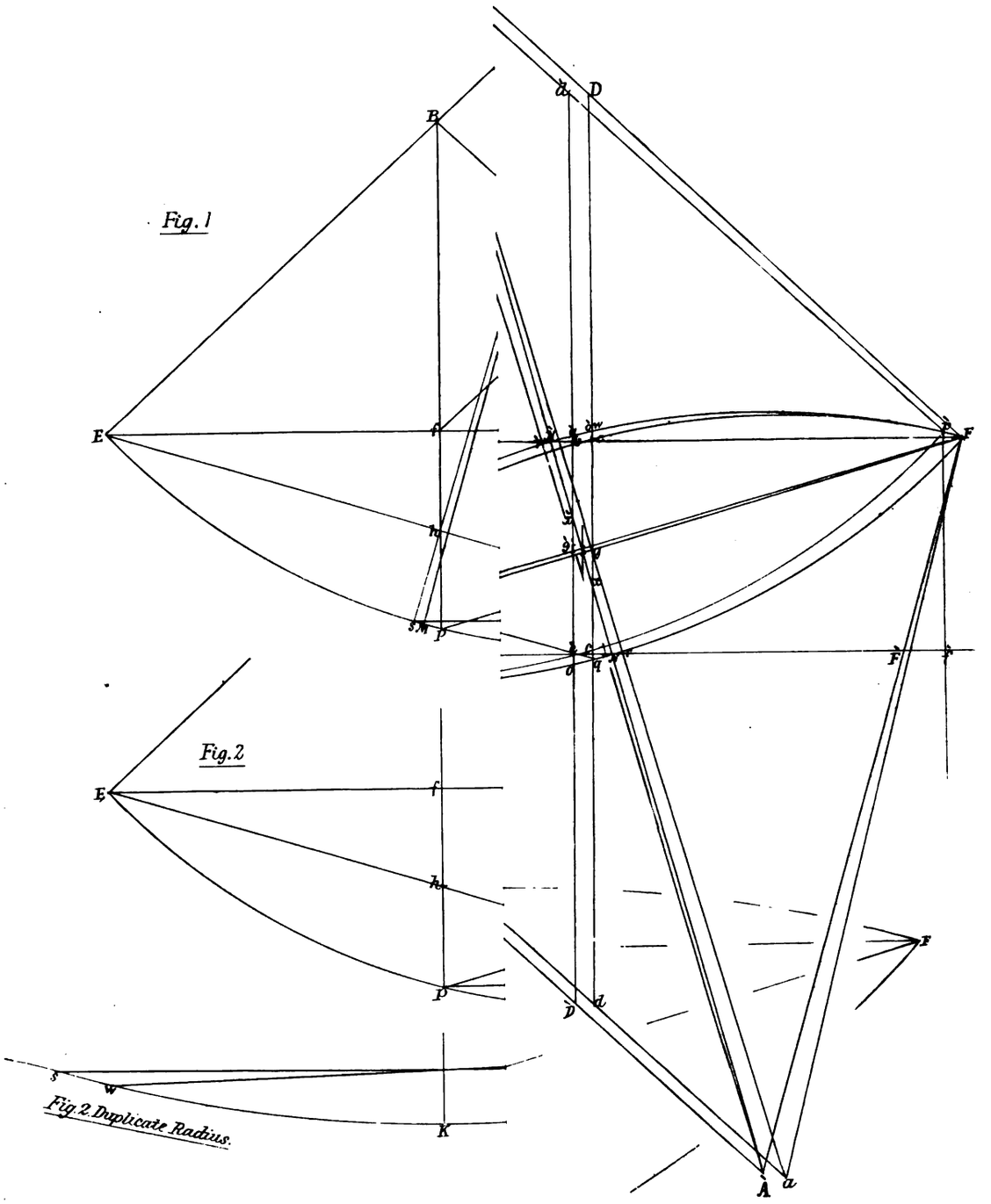


Fig. 1
Fig. 2
Fig. 2 Duplicate Radius.
Fig. 3
Enlarged Sections of arc q.r (Fig 1) & of arc q.t (Fig 2)
Scale 6 to 1.

Join $'a\,M$ and $'a\,F$. With centre a and radius $a\,F$ describe the arc $F\,e\,p$, cutting the chord $F\,E$ in point e; and with centre $'a$ and radius $'a\,M$ describe the arc $M\,'e\,'p$, cutting the chord $M\,N$ (part of $M\,N\,'f$) * in point $'e$. In point d cut off one-third of the arm $a\,p$ of the angle $p\,a\,F$; and in point $'d$ cut off one-third of the arm $'a\,'p$ of the angle $M\,'a\,'p$. Produce the perpendicular $D\,q$ to join point d, and produce the corresponding perpendicular $'D\,'q$ to join point $'d$.† Then, because point g bisects the duple radius $A\,a$ and point z bisects the duple radius $A\,'A$, when the first point of bisection moves to the second—from g to z, the point at the extremity of the first necessarily moves through twice the distance to the point at the extremity of the second—from a to $'A$; so that, if perpendiculars to chord $F\,E$ be drawn through points $'A$ and a, and also through point z, the distance between the perpendiculars $'A$ and a must equal the distance between the lines $e\,q$ and $'q\,'e$, for, evidently, when duple radius $A\,a$ changes to duple radius $A\,'A$ and (so to speak) a moves to $'A$, d must move to $'D$, and point z must accordingly bisect the distance between the lines $e\,q$ and $'q\,'e$. Again, because point g also bisects chord $F\,p$, and point z also bisects chord $F\,M$, the arc $M\,p$ (which equals $N\,q$) equals twice the arc through which point g in the one radius is transferred to point z in the other, and therefore equals the *arc of transference* $g\,'g$. Hence it follows: *first*, that the section $'e\,c$ of the chord $M\,N$, contained between the perpendiculars $'q\,'e$ and $e\,q$, equals $c\,N$ the section at the extremity N of the same chord; and, *secondly*, that the section $'N\,N$ of the duple radius $A\,'A$ is divided into three equal parts in the points $'x$ and x (namely, the sections $'N\,'x$, $'x\,x$, $x\,N$), and also that one-third of the line $'c\,'e$ is cut off by the point $'x$, and one-third of the line $c\,e$ by the point x, in which points respectively the duple radius $A\,'A$ cuts those lines. (*See the enlarged section of fig.* 4.) Now, since $'e\,c$ equals $c\,N$, parts of the chord of the arc $M\,N$, arc $q\,o$ is evidently less

* $M\,N\,'f$, the production of chord $M\,N$, is two-thirds of the greater chord corresponding to $F\,E$, and responds to $F\,f$, which is two-thirds of the chord $F\,E$.

† It may save the time of the reader in the first instance, and may also serve to bring prominently under notice a characteristic in this mode of investigation, which will be found, we think, to possess analytical utility and value, to point out that by *transferring* (or *attaching*) the arc $p\,r\,F$, subtending the (upper) angle $p\,A\,F$ to the respondent (lower) angle $p\,a\,F$, we may then *turn over* the latter angle, together with the arc $p\,r\,F$, on the chord $p\,F$ *as an axis* and impose it upon the upper angle $p\,A\,F$ and upon the respondent (upper) arc $p\,'r\,F$; so that the two become coincident and identical. Thus the point a, at the extremity of the respondent radial line $g\,a$, would fall on point A, at the extremity of the primary radial line $g\,A$, and the line $g\,a$ itself become identical with the line $g\,A$. The point r, bisecting the lower arc, would fall on point $'r$, bisecting the upper arc, and the entire lower arc $F\,r\,p$ become identical with the upper arc $F\,'r\,p$. Similarly with respect to the angle $M\,A\,F$ and the arc $M\,N\,F$ subtending it, by *transferring* the arc to the respondent angle $M\,'A\,F$ we may then *turn over* the latter, on the chord $M\,F$ *as an axis*, and impose it upon the upper angle $M\,A\,F$ and upon the upper arc $M\,'N\,F$; so that, as before, the two would become coincident and identical. It is important and instructive to note that in each of these two instances the point at the extremity of the respondent radial line falls upon the same point at the extremity of the primary radial line; that is to say, both the points a and $'A$ fall upon the same point A of the primary. If we go further than this, as indicated in the figure, and draw a third duple radius $'A\,'a$ through point $'g$ (which bisects the perpendicular $'e\,'q$) then the point $'A$, which is respondent and secondary with regard to the first and second, becomes primary with respect to the third and fourth. To proceed beyond the third, however, would be to virtually move the whole figure into a fresh position.

than arc $q\,N$, for $q\,N$ is the more distant part of that arc from the middle point K thereof. Also the part $F\,e$ of the upper arc $F\,e\,p$ responds to and manifestly equals the part $p\,q$ of the lower arc $F\,q\,p$, for both have the chord $F\,g\,p$ in common and are described respectively with the equal radii $a\,F$ and $A\,F$; therefore arc $F\,e$ equals arc $p\,q$.

And again, because arc $F\,q\,p$ is a part of arc $F\,N\,M$, arc $F\,e\,p$ has the same curvature as arc $F`N\,M$, and is the same as a part of the latter when the centre of description thereof is transferred from $a$ to $`A$ (the radii $a\,F$ and $`A\,F$ being equal). Hence, if with centre $F$ and radius $F\,e$ the arc $F`N\,M$ be cut in point w, arc $F\,w$ must equal arc $q\,p$, and arc $w`q$ must equal arc $p\,M$=arc $`q\,`N$=arc $q\,N$. Wherefore, since arcs $q\,N$ and $p\,M$ are equal and the distances $E\,f$, $f\,e$, $e\,F$ (in the chord $E\,F$) are equal, it follows that the three arcs $E\,M$, $M\,N$, $N\,F$, are equal each to each.

Note:—With regard to the three preceding demonstrations, it is not because we have any doubt as to the sufficiency and unobjectionable character of the first (Demonstration *a*) that the other two are supplied. All three, as demonstrations, are dependent upon the principle of trisecting the *intervening* arc (Fig. 3), of which the demonstration is contained in and forms a a part of the primary demonstration to Prop. IV.

In the second (demonstration *b*) the mode of applying the principle to the trisection of the arc subtending the given angle, differs from that adopted in the first, in such wise as to assist the explanation and demonstration (now to follow) of the most advantageous manner of applying the principle of the second method of trisection to the polysection of the arc and the angle.

The purpose of Demonstration *c* is to make clear the general *Rationale* of the second method of trisection, and also to exhibit, by means of the respondent arcs and angles, certain of the important structural correlations belonging to the circle. Plate 6 (opposite) repeats the Figs. of Plates 4 and 5, with some modifications, which will explain themselves. Figs. 4 and 5 of Plate 6 separate the compound Fig. 4 (of Plate 5) into the two simple figures.

Polysection of the Angle.

By the Second Method of Trisection.

In the latter part of the Introductory Note (page 95) we pointed out that the essential basis of the "Second Method" is the trisection of the connecting arc. An important feature, however, in the constructions belonging to Prop. IV., is the mode of determining the connecting arc by drawing a radius through the point of bisection of the perpendicular ($e\,q$ or $f\,p$) drawn from one of the extremities of the middle third part of the entire arc's chord.

Now this mode of determining the connecting arc,* although with regard to the *trisection* only of the angle by no means essential—for, as we have shown in Demonstration *b*, the *inter-*

* See definition of *connecting-arc*, page 101. The *intervening-arc* of Demonstration *b* equals one connecting-arc and a-half. Consequently the radius $A\,r$ drawn through the point of bisection (g) of the perpendicular, which bisects the *intervening-arc* of Demonstration *b*, cuts off three-fourths of a whole *connecting-arc*.

vening arc may be also readily determined by taking one of the extremities of the entire arc for a centre of description and a radius equal to one-third of the chord—becomes, when we proceed to apply the method to an equal division of the arc (or angle) into a greater uneven number of parts than three, such as the *pentasection*, *heptasection*, and so on, far the most convenient and advantageous; and, when, proceeding further still, we apply the method *generally* to the polysection of the arc, it becomes quite essential—(that is to say, it is not at present apparent to us that any other *general* mode of determining the connecting arc is practicable).*

For the purpose of the Polysection it will be requisite to show that the law of definite section of the arc by the radius drawn through the point bisecting the perpendicular, by which, as shown in the demonstration to Prop. IV., three-fourths of the *connecting arc* is cut off on the side nearest to the middle of the arc, is true, not only when that perpendicular to the chord is drawn from one of the extremities of the middle one-third part thereof, but also when drawn from one of the extremities of the middle divisional part of the chord when the chord is divided into any uneven number of equal parts greater than three. This we shall now proceed to do formally as a Theorem; and it may be here observed that the demonstration of this Theorem will virtually contain the demonstration of the Polysection, merely requiring to be directly applied to the latter by reference afterwards. To abbreviate and render more definite the statement of the Theorem we will precede it by the following special definition of the terms *divisional* and *connecting* arc.

DEFINITION.—When in an arc, several arcs, together less than the entire arc, are cut off and separated from each other by divisional points, so that curvilineal magnitudes intervene between the extremities of the adjacent arcs, the intervening curvilineal magnitude (or intervening fractional arc) is called a *connecting arc;* and the arcs, separated by the divisional points and the connecting arcs intervening between them, are called *divisional arcs.* (That section of the entire arc, compounded by one divisional arc and one connecting arc together, is called a *sectional arc.*)

PROP. V. THEOREM.

If the chord of an arc be divided into an *uneven* number of equal parts, and the chord be also bisected, and from the first divisional point, on either side of the point of bisection, a line perpendicular to the chord be drawn to intercept the arc, and through the point of bisection of that line a radius be drawn; and if the arc be divided into equal divisional arcs together less than the entire arc, in such wise that the number of the divisional arcs is the same as the number of the equal divisional parts of the chord, and the chord of each divisional arc is equal to each of the divi-

* It is probable that, for any individual case of division into a required number of parts, one or more other modes of determining the connecting-arc may be made available; but, if so, their use will be found, we think, to considerably complicate the demonstration, and will not be applicable to the *general* problem of *polysection*.

sional parts of the entire arc's chord, and so arranged and separated by the divisional points that (1) the point of bisection of the middle divisional arc coincides with the point of bisection of the entire arc, that (2) the connecting arcs intervening between them are equal each to each, and that (3) a half-connecting arc is at each extremity of the entire arc (namely, that each connecting arc between the adjacent ends of each two divisional arcs is equal to each of the other similar connecting arcs, and the two equal half-connecting arcs, at the two ends respectively of the entire arc, are together equal to each of the whole connecting arcs), then that part of the connecting arc contained between the point intercepted by the perpendicular and the point intercepted by the radius is three-fourths of the whole of that connecting arc, namely, of that connecting arc which intervenes between the extremity of the middle divisional arc and the adjacent extremity of the divisional arc next to it.

(Plate 7, Fig. 5). Let arc $E K F$, bisected in point K and subtending the angle $E A F$, be the given arc; and (1) Let the chord $E F$ of the arc be divided into five equal parts in the points a, b, c, d, and let the chord $E F$ also be bisected in the point k. From point b (next to k) draw $b q$ perpendicular to $E F$ and intercepting the given arc in point q, and from point c (next to k on the opposite side) draw the perpendicular $c p$ intercepting the given arc in point p. Bisect $b q$ in point g, and through g draw the radius $A r$. Then the arc $q r$ must be three-fourths of the whole connecting arc which intervenes between the extremity q of the divisional arc $p q$ (of which the chord equals $c b$) and the extremity adjacent to it of the next equal divisional arc between q and F.

Bisect the perpendicular $c p$ in point h, and through h draw the radius $A s$. From arc $r q$ cut off one-third in the point N, and from arc $s p$ cut off one-third in the point M (Prop. IV.). With centre N and distance $N F$ cut arc $M E$ in point L, and with centre M and distance $M E$ cut arc $N F$ in point O. Join $O M$, $O L$, $L F$. Then since point K bisects the entire arc $E K F$, and point N bisects arc $L F$, arc $E L$ equals twice arc $K N$, that is, equals arc $M N$; and, therefore, also, the chord $L F$ is at right angles to the radius $A N$ which intersects and bisects it. Again, since arc $O F$ manifestly equals arc $L E$, arc $L K O$ is bisected in point K, and point N must bisect an arc of which $N O$ is the one-half and which is less than arc $L K O$ by an arc equal to twice arc $K N$ (that is, equal to arc $M N$). Now, by the construction, arc $N O$ equals arc $M L$, and since point N also bisects arc $L F$, and arc $F O$ equals arc $E L$ (which equals arc $M N$), when a chord parallel to chord $F L$ is drawn from point O across radius $A N$, the opposite extremity of that chord must intercept the arc in point M, and the chord so drawn must be at right angles to the radius $A N$.* Therefore arc

* For let the chord be not at right angles to $A N$; then arc $M N$ must be either greater or less than arc $N O$. Let it be greater than arc $N O$, then must arc $M L$ be less than arc $N O$. Let it be

$L\,M$ is the arc taken from arc $L\,O$, arc $M\,O$ is the arc which remains, and arcs $L\,M$ and $N\,O$ are each equal to arc $M\,N$. Hence it follows that each of the arcs $E\,L$, $L\,M$, $N\,O$, $O\,F$, contains a divisional arc equal to the divisional arc $p\,q$ and two half-connecting arcs, each of which equals the half-connecting arc $q\,N$. Wherefore, since arc $q\,N$ is two-thirds of arc $q\,r$, $q\,r$ is three-fourths of the whole connecting arc $q\,w$, which intervenes between the extremity q of the divisional arc $p\,q$ and the adjacent extremity w of the next divisional arc $w\,v$. (For the arc $N\,O$ has been shown to contain a divisional arc [$w\,v$] equal to $p\,q$, and two half-connecting arcs [$N\,w$ and $v\,O$] each equal to $q\,N$.)*

(2.) Let the chord $E\,F$ of the arc be divided into seven equal parts in the points a, b, c, d, e, f (*Fig.* 6), and let the chord $E\,F$ also be bisected in the point k. From point c (next to k) draw $c\,q$, perpendicular to $E\,F$, intercepting the given arc in point q, and from point d (next to k, on the opposite side) draw the perpendicular $d\,p$, intercepting the given arc in point p. Bisect the perpendicular $c\,q$ in point g, and through g draw the radius $A\,r$. Then the arc $q\,r$ must be three-fourths of the whole connecting arc which intervenes between the extremity q of the middle divisional arc $p\,q$ and the adjacent extremity of the next equal divisional arc between the points q and f.

Bisect the perpendicular $d\,p$ in the point h, and through h draw the radius $A\,s$. From the arc $r\,q$ cut off one-third in the point N, and from the arc $s\,p$ cut off one-third in the point M (*Prop.* IV.). With centre N and distance $N\,F$ cut arc $N\,E$ in point J; and with centre M and distance $M\,E$ cut arc $M\,F$ in point P. Join $F\,J$; and from P parallel to $F\,J$ draw $P\,L$, intercepting arc $P\,J$ in point L. Then, because point K bisects the entire arc $E\,K\,F$, and point N bisects arc $F\,J$, arc $E\,J$ equals twice arc $K\,N$ (that is, equals arc $M\,N$). Since arc $F\,P$ manifestly equals arc $E\,J$, arc $J\,K\,P$ is bisected in point K, and point N bisects an arc of which $N\,P$ is the half, and which remains when an arc equal to arc $M\,N$ is taken from arc $J\,P$. Now, by the construction, arc $N\,P$ equals arc $N\,L$, therefore arc $L\,N\,P$ is the arc which remains, and arc $J\,L$ equals arc $M\,N$. From point L, through $A\,K$ at right angles, draw the chord $L\,O$. Then arc $P\,O$ equals arc $J\,L$ (because point K bisects arc $L\,O$), and point N bisects an arc of which arc $N\,O$ is the half, and which remains when an arc equal to arc $N\,O$ is taken from arc $L\,O$. But, by the construction, arc $N\,O$ equals arc $M\,L$, therefore arc

less, then arc $M\,L$ must be greater than arc $N\,O$. But arcs $M\,L$ and $N\,O$ are by the construction manifestly equal, and therefore arc $M\,N$ cannot be either greater or less than arc $N\,O$.

* With centre r and distance $r\,F$ cut arc $K\,E$ in point t. Then since point N bisects arc $F\,L$ and point r bisects arc $F\,t$, arc $t\,L$ equals twice arc $r\,N$, that is, equals arc $q\,N$. Also, if, with centre q and distance $q\,F$, arc $L\,E$ be cut in point z, arc $z\,L$ must equal twice arc $q\,N$, that is, must equal two half connecting arcs; and arc $E\,z$ must equal the divisional arc $p\,q$, and so on.

M O is the arc which remains, and each of arcs *L M* and *N O* equals arc *M N*. Hence it follows that of the arcs *E J*, *J L*, *L M*, *N O*, *O P*, *P F*, each contains a divisional arc, equal to the divisional arc *p q*, and two half-connecting arcs, each of them equal to the half-connecting arc *q N*. Wherefore, since arc *q N* is two-thirds of arc *q r*, the arc *q r* is three-fourths of the whole connecting arc which intervenes between the extremity (*q*) of the middle divisional arc *p q*, and the adjacent extremity (*w*) of the next divisional arc between the points *q* and *F*.

(3.) In the like manner it may be shown that the Theorem holds good when the chord of the given arc is divided into 9, 13, 15, 17, or any required uneven number of equal parts.

POLYSECTION OF THE ANGLE.

(By the Second Method of Trisection.)

PROP. VI. PROBLEM.

To divide a given angle into any required number of angles equal each to each.

(1.) Let the required number be five. (Fig. 7, Plate 8.)

Let *E A F* be the given angle. Describe the arc *E K F*, subtending the given angle, and draw the chord *E F* thereof, and divide the chord *E F* into five equal parts in the points *a*, *b*, *c*, *d* (*Prop.* 4, *P.F.*). From points *b* and *c*, draw perpendiculars to the chord intercepting the arc *E K F* in the points *q* and *p* respectively. Bisect the perpendicular *b q* in point *g*, and *c p* in point *h*, and through points *g* and *h* respectively draw the radial lines *A r* and *A s*. From *F* to *q* draw the chord *F q*; with centre *r* and distance equal to *q F* cut the arc *q E* in point *t*, and draw the equal chord *t r* cutting chord *F q* in point *o*. From *o r* (the terminal extremity of chord *t r*) cut off one-third in point *n*, and through point *n* draw the radial line *A N*. Bisect arc *N F* in point *O*. With centre *K* and distance *K N* cut arc *N E* in *M*, and bisect arc *M E* in *L*. Join *A L*, *A M*, *A O*. The given angle *E A F* shall be pentasected in the four radial lines *A L*, *A M*, *A N*, *A O*.

Because the chord of the arc *q p* equals (*b c*) one of the five equal divisions of the chord *F E*, and the radius *A N* cuts off two-thirds of arc *q r*, arc *q p* is one of five equal divisional arcs and arc *q N* the one-half of one of five equal connecting arcs contained in the entire arc *E K F*. (*Theorem*, Prop. V.) But arc *p M* equals arc *q N*, and therefore arc *M N* is one of five equal sectional arcs contained in arc *E K F*. Now, by the Construction and by Prop. V., arcs *E L*, *L M*, *N O*, *O F*, are equal each to each,

Fig. 5

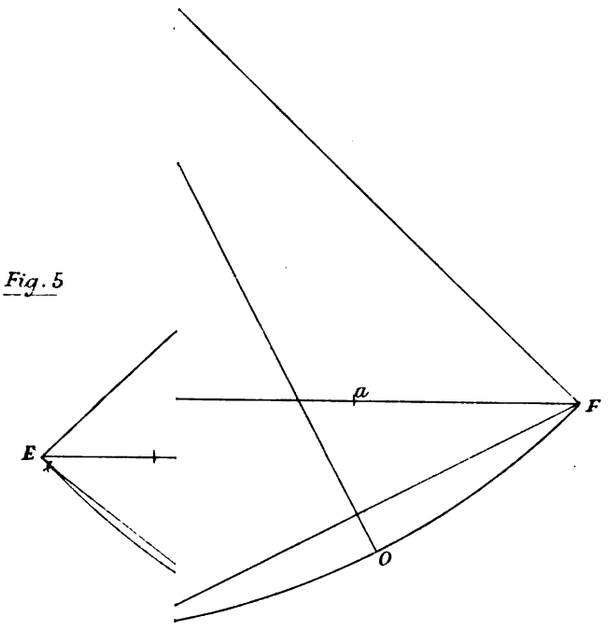
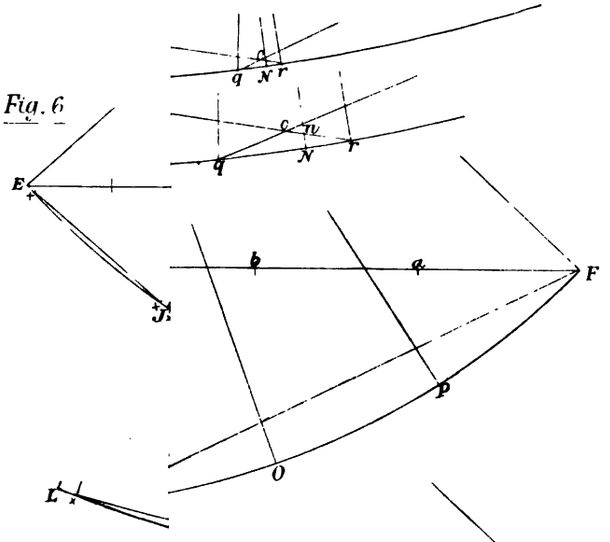
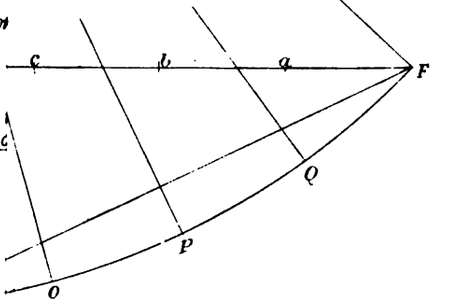


Fig. 6



Enlarged Section of



Obs. In each case

and each equals arc MN. Wherefore the five angles EAL, LAM, MAN, NAO, OAF, are equal each to each.

(2.) Let the required number be six (*Fig.* 1, Plate 4).

Trisect the arc subtending the given angle (Prop. IV.). Bisect each of the three equal arcs and to each of the five divisional points, thus obtained, draw a line from the vertex of the given angle. The given angle will then be divided into six angles equal each to each.

(3.) Let the required number be ten (*Fig.* 7, Plate 8).

Pentasect the arc subtending the given angle (*Section* 1). Bisect each of the five equal arcs, and from each of the nine divisional points thus obtained draw a radius to the vertex of the given angle. The given angle will then be divided into ten angles equal each to each.

(4.) Let the required number be seven (*Fig.* 8).

Divide the chord EF, of the arc EKF subtending the given angle EAF, into seven equal parts in the points a, b, c, d, e, f (Prop. 4, *P.F.*). From points c and d draw perpendiculars to the chord intercepting the arc in points q and p respectively. Bisect the perpendiculars cq and dp in points g and h respectively. Through point g draw the radius Ar and through point h the radius As. From arc qr cut off two-thirds in point N, and from arc ps also cut off two-thirds in point M (Prop. IV.). Join AM and AN. Then, by the *Theorem* Prop. V., the arc MN is one of seven arcs, equal each to each, contained in the arc EKF subtending the given angle, and therefore the angle MAN is one of seven angles, equal each to each, contained in the given angle EAF.

(5.) Let the required number be nine (*Fig.* 9).

Divide the chord EF, of the arc EKF subtending the given angle EAF, into nine equal parts in the points a, b, c, d, e, f, j, l, (*Prop.* 4, *P.F.*) From points d and e draw perpendiculars to the chord, intercepting the arc in points q and p respectively. Bisect the perpendiculars dq and ep in points g and h respectively. Through point g draw the radius Ar, and through point h the radius As. From arc qr cut off two-thirds in point N, and from arc ps also cut off two-thirds in point M (Prop. IV.). Join AM and AN. Then, by *Theorem*, Prop. V., the arc MN is one of nine arcs, equal each to each, contained in the arc EKF subtending the given angle; and therefore the angle MAN is one of nine angles, equal each to each, contained in the given angle EAF.

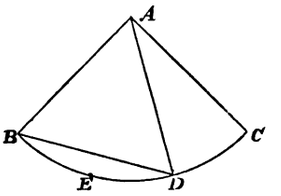
(6.) In the like manner (and by bisection or repeated bisection) a given arc or a given angle may be divided into any required number of arcs or angles respectively equal each to each.

The following problems may suffice to illustrate the application of "Trisection and Polysection of the Angle." In these and other cases either of the two distinct methods demonstrated may be chosen. The First Method, however, will perhaps be found the most simple and convenient for ordinary purposes of polysection.

PROP. IX. PROBLEM.

To describe an equilateral triangle on a given straight line.

Let $A\,B$ be the given straight line. From point A, at right angles to $A\,B$, draw $A\,C$ equal to $A\,B$. With centre A and radius $A\,B$ describe the arc $B\,C$. Trisect the arc $B\,C$ in points D and E. Join $A\,D$, and draw the chord $B\,D$ (of the arc $B\,D$).



Then, because the sides $A\,B$ and $A\,D$ are equal and the third side $B\,D$ is at right angles to the line of bisection $A\,E$, the angles $A\,B\,D$ and $A\,D\,B$ are equal. But the three angles in a triangle are together equal to two right angles, and the angle $B\,A\,D$ is two-thirds of a right angle. Therefore each of the other two angles is also two-thirds of a right angle. Wherefore the triangle $A\,B\,D$, described upon the given straight line $A\,B$, is equiangular and equilateral.

PROP. X. PROBLEM.

To inscribe in a given circle an equilateral and equiangular pentagon.

(Plate 9, Fig. 1.) In the given circle $A\,D\,F\,B\,E$ inscribe the eq. triangle $A\,D\,B$. Bisect the sector $A\,C\,B\,E$ in the line $C\,E$, and divide the arc $E\,B$ into five equal parts in the points 1, 2, 3, 4 (Prop. III. or VI.). Make the adjacent arc $B\,a$ equal to each of the five divisional parts of arc $E\,B$, and add arc $B\,a$ to arc $E\,B$. Draw the chord $E\,a$ (of arc $E\,a$), and draw the chords $a\,b$, $b\,d$, $d\,e$, $e\,E$, each equal to the chord $E\,a$. The five-sided polygon so inscribed shall be the required regular pentagon.

Because the inscribed triangle $A\,D\,B$ is equilateral, the arc $E\,B$ (one-half of arc $A\,B$) is one-sixth of the circle's circumference. But arc $E\,a$ is greater than arc $E\,B$ by one-fifth of $E\,B$. Therefore arc $E\,a$ is one-fifth of the circle's circumference. And, since the other four chords are each equal to the chord $E\,a$, the polygon of which the five chords are the five sides is equilateral. Wherefore an equilateral and equiangular pentagon has been inscribed in the given circle.

PROP. XI. PROBLEM.

To inscribe in a given circle an equilateral and equiangular heptagon.

(Plate 9, Fig. 2.) Divide the circumference of the given circle $A\,H\,B\,E$ into eight equal parts in the points $A\,D\,E\,F\,B\,G\,H\,K$ by bisection. Divide the arc $D\,E$ into seven equal parts (Prop. III. or VI.). Make the adjacent arc $E\,a$ equal to one of the seven parts of arc $D\,E$, and add arc $E\,a$ to arc $D\,E$. Draw the chord $D\,a$ (of arc $D\,a$), and draw the chords $a\,b$, $b\,c$, $c\,d$, $d\,e$, $e\,f$, $f\,D$, each equal to chord $D\,a$. The seven-sided polygon so inscribed shall be the regular heptagon required. (The demonstration of this proposition is virtually the same as that of the preceding prop.)

PROP. XII. PROBLEM.

To inscribe in a given circle an equilateral and equiangular nonagon.

(Fig. 3.) Divide the circumference of the given circle $E\,b\,d\,g$ into five equal parts in the points E, F, 4, 6, 8 (Prop. X.). Bisect arc $E\,F$ in point K, and bisect also each of the other four divisional arcs. The circle's circumference is then divided into ten equal parts, of which the arc $E\,K$ is one. Divide arc $E\,K$ into nine equal parts (Prop. VI. or III.). Make the adjacent arc $K\,a$ equal to each of the nine equal parts of $E\,K$, and add arc $K\,a$ to arc $E\,K$. Draw the chord $E\,a$ (of arc $E\,a$), and draw the chords $a\,b$, $b\,c$, $c\,d$, $d\,e$, $e\,f$, $f\,g$, $g\,h$, $h\,E$, each equal to the chord $E\,a$. The nine-sided polygon so constructed and inscribed shall be the regular nonagon required. (The demonstration of this proposition is virtually contained in that of Prop. X.).

PROP. XIII. PROBLEM.

To inscribe a regular polygon of eleven sides in a circle.

(Fig. 4.) Let $A\,E\,F\,B\,G\,D$ be the given circle in which it is required to inscribe the regular polygon of eleven sides.

Find the centre C of the circle; and through C draw any diameter $A\,B$. With centre A and radius $A\,C$, cut the circumference of the given circle in the points D and E. With centre B and radius $B\,C$, cut the circumference of the given circle in the points G and F. Through C, the centre of the circle, join $F\,D$ and $E\,G$; thus dividing the given circle into six equal angles in the lines $C\,A$, $C\,E$, $C\,F$, $C\,B$, $C\,G$, $C\,D$ (Euclid IV. 15). Bisect the divisional angle $E\,C\,F$ in the line $C\,K$. Divide the half-angle $K\,C\,F$ into eleven equal parts (Prop. VI. or III.) and make the adjacent angle

$F C b$ equal to each of the eleven divisional angles. Join the angle $F C b$ to the angle $K C F$, and draw the chord $K b$ (of the arc $K b$). Then, because the sector $K C F$ is the one-twelfth of the entire circle, and the lesser sector $F C b$ equals one-eleventh of the greater sector $K C F$, it follows that the sector $K C b$, compounded of the two sectors when united, is the one-eleventh part of the entire circle. Draw therefore ten other chords, each equal to the chord $K b$. The polygon of eleven sides so constructed and inscribed in the given circle is manifestly the regular, equilateral and equiangular, hendecagon required.

PROP. XIV. PROBLEM.

To inscribe a regular polygon of Thirteen Sides in a Circle.

(Fig. 5.) Let $a\ d\ g\ m$ be the given circle. It is required to inscribe a regular polygon of thirteen sides in the circle $a\ d\ g\ m$.

Divide the given circle into seven equal angles having the centre C of the circle for their common vertex (Prop. XI.); and let $5\ C o$ be one of the seven equal angles. Bisect the angle $5\ C o$ in the line $C K$. Divide the angle $K C o$ into thirteen equal angles (Prop. VI.), and make the adjacent angle $K C a$ equal to one of the thirteen equal angles. To the angle $o C K$ add the angle $K C a$. With centre a and distance $a o$ cut the circle's circumference in b. With centre b and same distance cut the circumference in c. With the same distance in each case and the successive divisional points as centres, cut the circumference successively in the points $d, e, f, g, h, i, k, m, n$. The circle's circumference shall be divided into thirteen equal parts in the thirteen equidistant divisional points, namely, the points $a, b, c, d, e, f, g, h, i, k, m, n, o$. Join each point to the next by drawing the chord. The thirteen equal chords shall be the equal sides of the inscribed polygon required.

Because the angle $o C K$ is the one-fourteenth part of the circle; and the angle $o C a$ is greater than the angle $o C K$ by the one-thirteenth part thereof, the angle $o C a$ is manifestly the one-thirteenth part of the circle. Now, each one of the successive divisional spaces in the circumference is made equal to the arc $o a$ subtending the angle $o C a$. Wherefore the chords drawn from each successive divisional point to the next are the equal sides of an equilateral and equiangular polygon of thirteen sides inscribed in the given circle as required.

SCHOLIUM.—Euclid has shown (Bk. IV., Pr. 16) that the circumference of a circle may be evenly divided into fifteen equal parts. In a manner similar to that exhibited in the preceding examples, it can be shown that the circle's circumference may be divided into fifteen, seventeen, nineteen, twenty-one, twenty-three, and into any required number of equal parts, odd or even.

Plate 9.

Fig 1

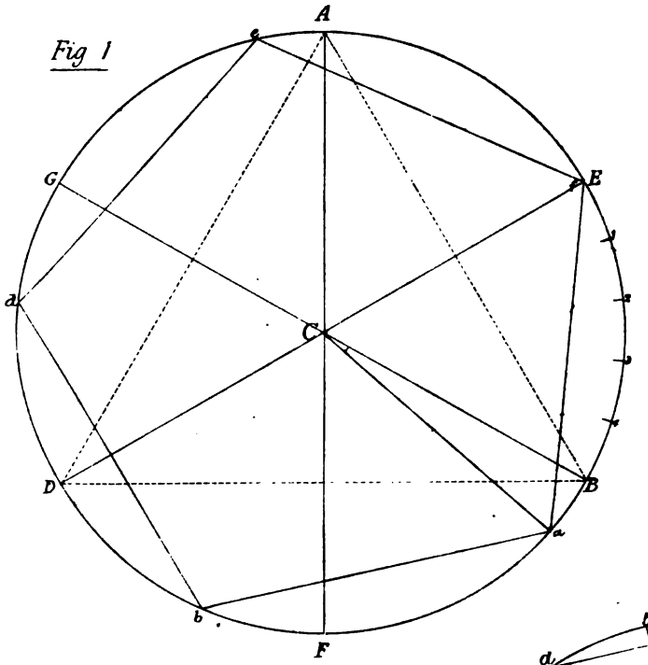


Fig. 2

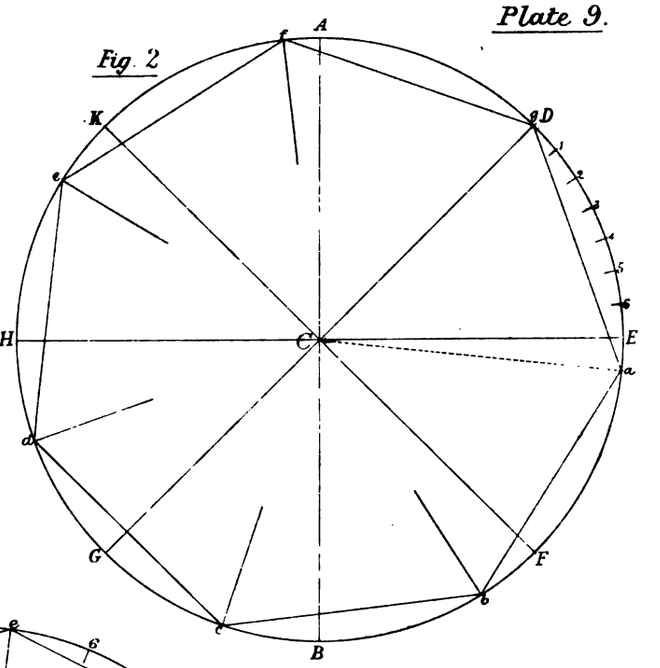


Fig. 3

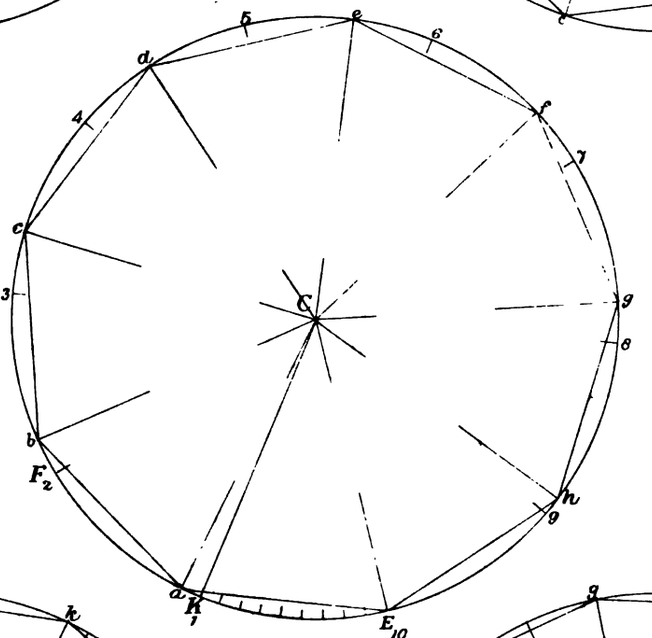


Fig. 4

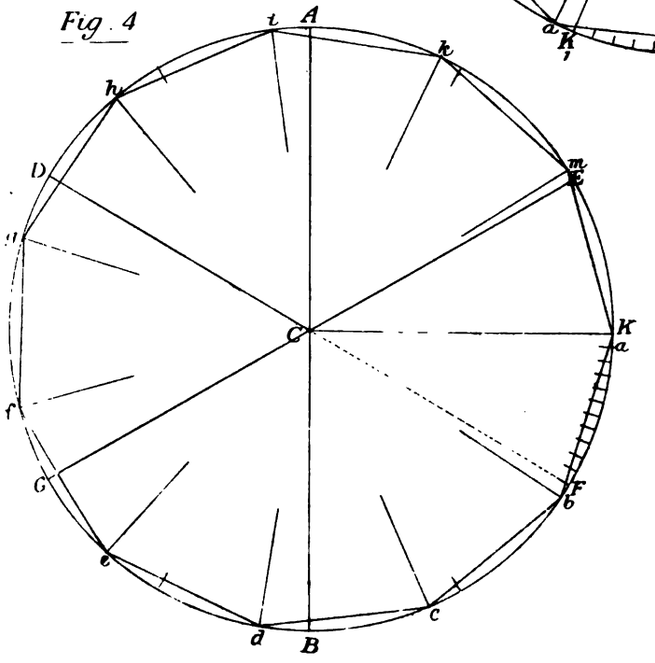
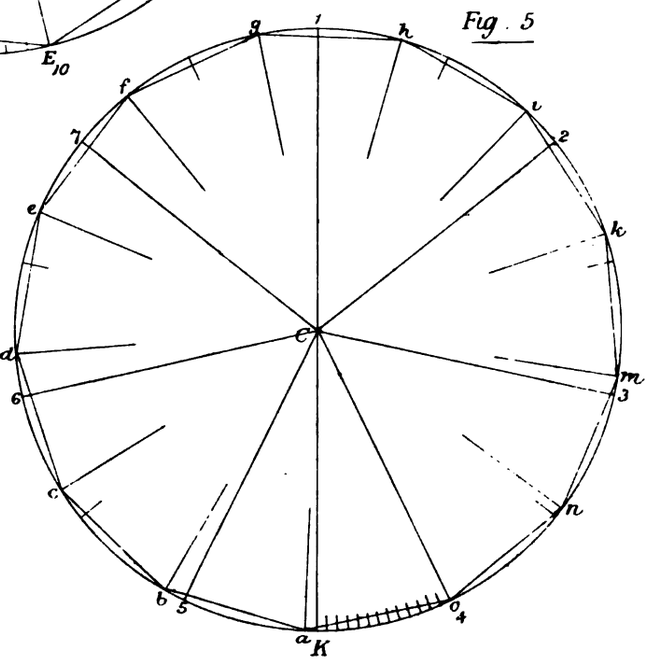


Fig. 5

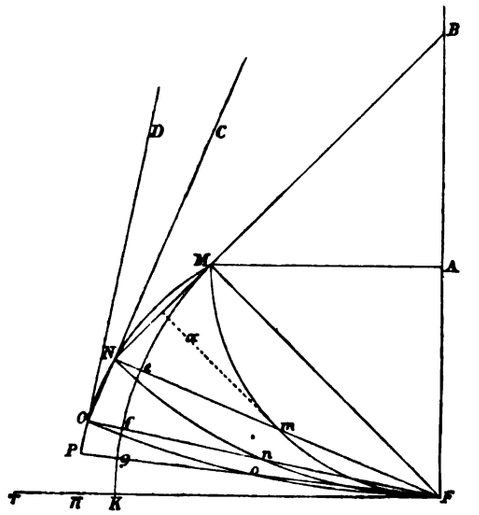


APPENDIX TO PART SECOND.

THE TRANSVERSE ARC OF THE PRIMARY CHORD

(Viz.: That of which the Chord of the Primary Arc is the Radius).

If, with the original point of a given primary arc for a centre and the chord of that arc for a radius, a transverse arc be described terminated by its interception of the tangent line belonging to the primary arc, then—(1.) If the primary arc be bisected and the chord of the half arc, next to the original point, drawn from the original point, be produced to intercept the transverse arc, the point of interception will bisect the transverse arc. (2.) If the half arc next the original point of the primary be bisected, and the chord of the quarter arc, drawn from the original point to the point of bisection, be produced to intercept the transverse arc, the point of interception will bisect the half of the transverse arc next the tangent line. (3.) If the quarter-arc, next the original point of the primary, be bisected, and the chord of the one-eighth arc, drawn from the original point, be produced to intercept the transverse arc, the point of interception will cut off one-eighth of the entire transverse arc next the tangent line. (4.) In like manner, if any part of th primary arc, greater or less than the half thereof, be cut off, and through the point of section the chord, drawn from the original point of the primary, be produced to intercept the transverse arc, the part of the transverse arc, next the tangent line of the primary, cut off by the interception of the produced chord, will correspond to the part of the primary arc cut off by the point of section through which the produced chord is drawn; that is to say, the sectional part of the transverse arc, next the tangent line, will be a fraction of the entire transverse arc, similar and respondent to the fraction of the entire primary arc cut off between the point of section through which the produced chord is drawn and the original point.



(Fig. 1.) Let the arc FmM, subtending the angle FAM, be the given primary arc. With centre F and radius FM describe the transverse arc MK, intercepting the tangent line FT in the point K. The arc MK is the *Transverse Arc of the Primary Chord.**

* The arc MNπ is the *Composite Transverse Arc of the Ultimate Chord;* which is compounded of the *Transverse Connecting Arcs* MN, NO, OP, &c., &c., *ad inf.* Each of the Trans. Conn. Arcs is Cyclal; but the Composite Trans. Curve, as a whole, is not Cyclal—*i.e.*, has no common centre of description.

Bisect the primary arc FM in the point m, and produce the chord Fm of the half arc Fm, through m, to intercept the transverse arc MK in the point e. Join Am. Then because the chord FM is perpendicular to the radius Am, and the chord Fm perpendicular to the radius which bisects the angle FAm, the line Fme bisects the transverse angle MFK. Bisect the half arc Fm in the point j, and through j, from F, draw Fjf intercepting the transverse arc MK in f. Join Aj. Then because radius Aj bisects the angle FAm and is perpendicular to the line Fme, and the line Fjf is perpendicular to the radius which bisects the angle FAj, the line Ff bisects the transverse angle eFK.

Wherefore it is made evident that whatever part of the primary arc be cut off by a chord-line drawn from its original point, if that chord-line be produced to intercept the transverse arc, the part of the transverse arc cut off thereby and contained between the point of interception and the tangent line of the primary, must have the same ratio to the whole transverse arc, which the part, cut off by the chord-line from the primary, has to the whole primary arc.

PART THIRD.

"QUADRATRIX" AND "LUNAR ANALYSIS OF THE CIRCLE;"

WITH LUNAR PROBLEMS.

THE QUADRATRIX OF DINOSTRATUS:—

We will first quote the account of the *Quadratrix* by Professor Wallace in the Encyclopædia Britannica.

("Squaring or Quadrature of the Circle")

"We learn from Simplicius that Nicomedes and Apollonius both attempted to square the circle, the former by means of a curve which he called the *quadratrix;* the invention of which, however, is ascribed to Dinostratus; and the latter, by the help of a curve, denominated the sister to the tortuous line, or *spiral*, and which was probably the quadratrix of Dinostratus; the nature of which and the manner of its application to the subject in question, we shall briefly explain.

"Let $A F B$ be a quadrant of a circle (Fig. 2), and C its centre; and conceive the radius $C F$ to revolve uniformly about C, from the position $C A$, until at last it coincides with $C B$; while at the same time a line $D G$ is carried with a uniform motion from A towards $C B$; the former line continuing always parallel to the latter until at last they coincide; both motions being supposed to begin and end at the same instant: The point in which the revolving radius $C F$ and the moveable line $D G$ intersect one another, will generate a certain curve line $A E H$, which is the *quadratrix* of Dinostratus.

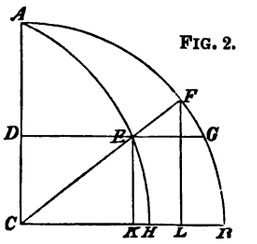


FIG. 2.

"Draw $E K$, $F L$, both perpendicular to $C B$; then because the radius $A C$, and the quadrantal arch $A F B$, are uniformly generated in the same time by the points D and F, the contemporaneous spaces described will have to one another the same ratio as the whole spaces; that is $A D : A F :: A C : A B$; hence we have $A C : A B :: D C$, or $E K, : F B$. Now, as the moveable point F approaches to B, the ratio of the straight line $E K$ to the arch $F B$ will approach to, and will manifestly be ultimately the same as, the ratio of the straight line $E K$ to the straight line $F L$, which again is equal to the ratio of $C E$ to $C F$; therefore the ratio of the radius $A C$ to the quadrantal arch $A F B$ is the limit of the ratio of $C E$ to $C F$, and consequently equal to the ratio of $C H$ to $C B$; H being the point in which the quadratrix meets $C B$. Since, therefore $C H : C B :: C A$, or $C B$, : quad. arch $A F B$, if by any means we could determine the point H, we might then find a straight line equal to the quadrantal arch (by finding a third proportional to $C H$ and $C B$), and consequently a straight line equal

to the circumference. The point H, however, cannot be determined by a geometrical construction, and therefore all the ingenuity evinced by the person who first thought of this method of rectifying the circle (which certainly is considerable), has been unavailing."

Finding ourselves, after carefully considering the exposition, unable to admit the soundness of the reasoning* by which the conclusion is arrived at that $CH : CB :: CA$, or CB, : quadrantal arc AFB, it occurred to us to try whether we could not directly resolve the arc AEH, and so determine the point H and the value of CH, notwithstanding the above somewhat discouraging assurance that it cannot be done. It soon appeared that the problem resolved itself, in the first place, into the question of whether the quadratrix curve AEH is or is not demonstrably a simple cyclal curve, *i.e.*, a curve having a centre of description common to every part of it.†

Repeating the figure on a somewhat larger scale (Fig. 3): Bisect the perpendicular radius AC in d. From d, at right angles to AC, draw dg, intersecting the quadratrix in g. Bisect the quadrantal arc AB in J. Join gJ, and join also dJ, cutting the quadratrix in f. From f, at right angles to AC, draw fa, intercepting AC in a. And through f, from C, draw the radius Cb, intercepting the quad. arc in b; also, join DF, cutting the quadratrix in g. Then, because (by the synchronal conditions of the case) ad is proportional to bJ in the same ratio as $Ad : AJ$, and as $AC : AB$, we have $af : fb :: dg : gJ :: df : fJ$. And since dD is also proportional to JF in the ratio of AC to AB, we have also $Dg : gF :: df : fJ$. And, finally, $dg : gJ :: df : fJ :: Dg : gF :: CH : HB$. Evidently, therefore, if the perpendicular AC and the arc AB be proportionately divided into any number of parts, so that each successive part of AB be proportional to the corresponding successive part of AC in the ratio of the whole of AB to the whole of AC, and then the successive divisional points in AC be directly joined to the corresponding successive points in the arc AB, that part of each connecting line, between the quadratrix curve and the perpendicular AC, will be proportional to the

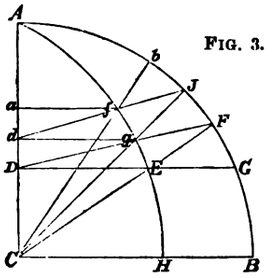


FIG. 3.

* So far from the assumption "Now as the moveable point F approaches to B, the ratio of the straight line EK to the arc FB will approach to and ultimately become the same as the ratio of the straight line EK to the straight line FL" being manifestly true, it seems to us to be manifestly untrue with regard to the latter part of it, for the sine of an arc and the arc itself can never become coincident, nor can the ratio of any fraction of the lesser sine EK to a similar fraction of the greater sine FL ever become equal to the ratio of the former to a similar fraction of the greater arc AFB.

† See Primary Definitions. No. 5, page 14.

other part of the connecting line, between the quadratrix and the arc AB, in the ratio of CH to HB; (as, for example $df:fJ$ and $Dg:gF$). Hence it follows (1) that the ratio of CH to HB is not, as Professor Wallace supposes, a derivative from and ultimate limit to the ratio of CE to EF, for it exists from the very commencement and remains constant throughout the description of the quadratrix. And (2) because the ratio (of CH to HB) remains constant from the commencement throughout the quadratrix curve, that curve is so uniformly related to the quadrantal arc AB that, as AB is a cyclal curve, so must the quadratrix be cyclal also; for, manifestly, if the curvature of AH were in one part greater than or equal to, and in another part less than, cyclal, it would be impossible that the parts of the connecting lines cut off by the quadratrix, on the one side of it, could have the same uniform constant ratio to the corresponding remaining parts of those lines, on the other side of it, throughout the curve. Wherefore it clearly appears that the quadratrix AH is a simple cyclal curve; *i. e.*, the arc of a circle.

Having determined the quadratrix curve AH to be cyclal, by application of Prop. 39 (*P.F.*), its centre of description may be at once determined. Accordingly (Fig. 4):—Draw the chord AH, and bisect it in the point n. Produce BC through C, and from n, at right angles to AH, draw nR, intercepting the production of BC in the point R. Then R is the centre of description and RA the radius of the arc (*quadratrix*) AH. The trigonometrical measurement of the radius, and determination of the value of CH, readily presents itself as follows:—Draw the chord Ag, of the half quadratrix, and bisect it in p; and, at right angles to AC, draw pq. Join pR cutting AC in its point of bisection d. Then since qp and qd are manifestly equal, dC (the half of AC) equals CR. Therefore AC equals twice RC and RA is the diagonal of the rectangle contained by AC and CR. Now radius RH is the same as RA; and $CH = RH - RC$. Let $AC = 1$. Then $RC = \cdot5$. RA (or RH) $= 1\cdot118034\ldots$ $CH = \cdot618034\ldots$ Let $CH = 1$. Then $CB = 1\cdot618034\ldots$* Hence, if the ratio $CB : CH$ equalled the ratio of the quadrantal arc

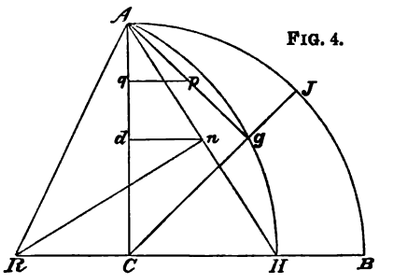


FIG. 4.

* We have here a very interesting numerical (quantitative) relationship. The greater radius $RA = \sqrt{1\cdot25} = 1\cdot118034\ldots$ From this deduct $\cdot5$. Then the prime number 1·0 is a mean proportional between the remainder and that remainder plus 1·0 (namely $\cdot618034 : 1\cdot0 :: 1\cdot0 : 1\cdot618034$), as appears above. In the same connexion it may be remarked as noteworthy that the angle subtended by the quadratrix is nearly a mean proportional between 45° and 90°. For the mean would be (about) 63°·64 and the actual angle is (about) 63°·44. This latter circumstance, however, is (pro-

to its radius, $(1{\cdot}61803 \times 2) = 3{\cdot}23606\ldots$ would be the ratio of the circle's circumference to the diameter as unity, but which is certainly much greater (by nearly one part in thirty-three) than the true ratio.

Although it now appears that the *quadratrix* does not possess that extraordinary property first attributed to it by its discoverer, Dinostratus, in ancient times, and also assigned to it by Prof. Wallace and other mathematicians up to the present time, we need not conclude that its actual relationship to the quadrantal arc is devoid of interest to geometricians. Before passing on, we will observe (1) that if a point be taken anywhere between C and R, for a centre of description, and, joining that point with A for a radius, an arc be described therewith terminated by the line $R\,C\,B$, that arc will be intermediate between the quadratrix and the quadrantal arc. Consequently, by taking any number of successive points, between C and R, a corresponding number of arcs may be described between the quadratrix and quadrantal arc—all of them having the point A in common (at the original extremity of each). In like manner, by taking a point as the centre of description further than R from C, an arc, having the same point A at its original extremity, may be described between the quadratrix and the perpendicular $A\,C$; and so, by taking successive points further than R from C, any number of arcs, having the point A in common, may be described between the quadratrix and the perpendicular $A\,C$. The limit to this process shows the perpendicular $A\,C$ to have a two-fold character, for, whilst it is the radius of the quadrantal arc, it may be also considered as the ultimate quadratrix when the centre of description, beyond R, becomes infinitely distant and the radius of infinite length: for then (if it were possible), no curvature remaining, the quadratrix would become coincident with the straight line. (2.) That the uniformity of the ratios, in the *parts* of the lines directly connecting the divisional points in the perpendicular radius $A\,C$ with the corresponding divisional points in the quadrantal arc $A\,B$, on each side respectively of the quadratrix, holds good with regard to each of the other intermediate arcs, whether nearer to or further from the quadrantal arc than the quadratrix itself, and for the same reason, namely—that in each of them the curvature is cyclal.

The angle subtended by the quadratrix=about $63^{\circ}{\cdot}435$. Since it may be considered as areally belonging either to the lesser circle—of which $C\,A$ is the radius, or to its own circle—of which $R\,A$ is the radius, the area of the figure contained by the right angle $A\,C\,B$ and the quadratrix may be expressed in terms of either radius. Thus:—as a part of the lesser circle (when $C\,A=1$) the area of the figure$=\cdot4422\ldots$; as a part of its own circle (when $R\,A=1$) its area$=\cdot35369\ldots$ [This is reckoning the angle, as stated, $63^{\circ}{\cdot}435$; and the circle's area as $3{\cdot}14269\ldots$]

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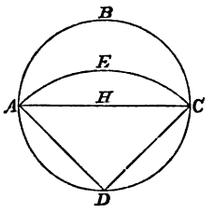
bably—so to speak) merely a coincidence, but the former is a fact belonging to the natural correlations of numbers, quantities and magnitudes.

THE LUNE OF HIPPOCRATES.

We will, in the first place, take (as before, by quotation from the "Encyclopædia Britannica") the brief explanation of the *Lune* by Professor Wallace of Edinburgh:—

"Hippocrates of Chios was likewise engaged in trying to resolve the same problem, and it was no doubt in the course of his inquiries into this subject, that he discovered the quadrature of the curvilineal space, which is now known by the name of the *Lune* of Hippocrates. The nature of this discovery may be briefly explained as follows. Let *A B C D* be a circle, *H* its centre, *A C* its diameter, *A D C* a triangle inscribed in the semicircle, having its sides *A D*, *D C* equal to one another. On *D* as a centre, with *D A* or *D C* as a radius, let the quadrantal arch *A E C* be described, then shall the curvilineal space bounded by the semicircle *A B C* and the quadrantal arch *A E C* (which is the *Lune* of Hippocrates) be equal to the rectilineal triangle *A D C*.*



"Although Hippocrates' discovery has led to no important conclusion either relating to the quadrature of the circle or that of any other curve, yet, at the time it was made, it might be regarded as of some consequence, chiefly because it showed the possibility of exhibiting a rectilinear figure equal to a space bounded by curved lines, a thing which we have reason to suppose was then done for the first time, and might have been fairly doubted, considering the insuperable difficulty that was found to attend the quadrature of the circle or its rectification."

We shall proceed to show that, had Hippocrates seen his way to continue the process and to compute the results at each stage, although he would not have arrived at the true geometrical ratio or rectification of the circle, he would have obtained the same result which mathematicians have got at by the Archimedean method of inscribing and circumscribing polygons, and, if he had succeeded, by a succession of *lunes*, in obtaining that result, he might have, and very probably would have, been just as certain as mathematicians now are that the veritable ratio or rectification of the circle had been achieved, and by a method which he perhaps would then have called *the lunar approximative* process.

Repeating the figure on a larger scale, at Fig. 2, Plate 10 and Plate 11;—Let *A B* C_2*D* be a circle; of which *C* is the centre, and *A* C_2, *B D* diameters, at right

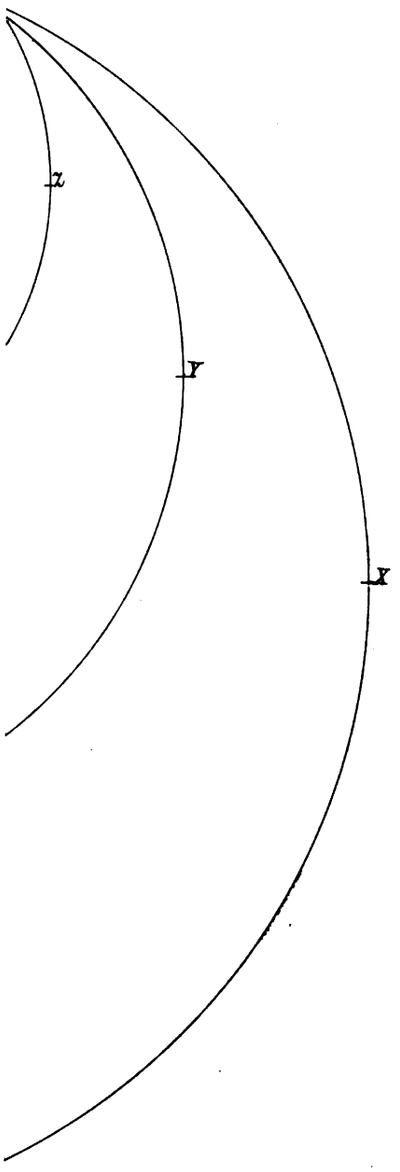
* Because the diameter *A C* is the diagonal of a square of which the chords *A D* and *C D* are two of the sides, the quadrantal segment *A C E* is twice the areal magnitude of each of the lesser quadrantal segments *A D* and *D C*, and therefore equals the sum of their magnitudes. It follows that the *Lune* equals the triangle. See also *Fig.* 1, Plate 10.

angles to each other. Bisect the quadrants BA and AD in the points E and F and the octants BE and EA in the points H and I, and draw the respective chords. Draw also the chords C_2B and C_2D of the two quadrants in the semicircle BC_2D. With centre C_2, and radius C_2B describe the circle BDS, of which the quadrantal arc BA_2D is a part. Bisect the octants BA_2 and DA_2 in the points e and f, and draw the chords of the octants and half-octants. From A_2, at right angles to A_2B, draw A_2m equal to the chord A_2B. Bisect the distance mB in n; and with centre C and radius equal to mn, cut the diameter BD in the points b and d. Draw the chord of the quadrant ST in the second circle, and with centre b and radius equal to the chord ST, cut the perpendicular diameter A_2S in the point C_3.* With centre C_3 and radius C_3b, describe the circle $aRQY$, of which the octantal arc bad is a part. (Plate 11.) Draw the chord ab of the half-octant, and from a at right angles to ab draw ap equal to ab. Bisect the distance pb in q; and with centre C and radius equal to pq, cut the diameter in the points β and δ. Draw the chord of the quadrant RQ in the third circle, and with centre β and a radius equal to chord RQ, cut the perpendicular diameter aQ in the point C_4. With centre C_4 and radius $C_4\beta$, describe the circle $aPO\mathcal{X}$, of which the half-octantal arc $\beta a\delta$ is a part. And so proceed in like manner. The centre of description for the next circle, in the perpendicular diameter aO, will be the point C_5, and the *lunar* arc $(\gamma\epsilon)$, belonging to it, will be a quarter octant; and so on.

Since the diameter of a circle is to the quadrant's chord (or the quadrant's chord to the radius) as the diagonal to the side of a square, and the square on a diagonal has twice the area of the first square (Prop. 25, *P.F.*); and since areal circles are proportional to each other as the squares on their diameters (Prop. 38, *P.F.*); and because the similar parts of areal circles—such as quadrantal segments, octantal segments, and so on—are to each other as the circles of which they are the similar parts; it follows that the quadrantal segment BA_2D is areally twice the quadrantal segment BEA; and the octantal segment BA_2 is twice the octantal segment BE. But the octantal segment bad is twice the octantal segment BA_2, and therefore the octantal segment bad is four times the octantal segment BE; that is to say, the octantal segment bad equals the two octantal segments in the quadrant BA_2D, and equals the four octantal segments in the semicircle BAD. In like manner, the half-octantal segment $\beta a\delta$, described (from centre C_4) with the radius of the next circle of duplicate area, is twice the half-octantal segment bka; and the half-octantal segment bka is

* Or thus: Bisect the half quadrant BA_2 in e, and join eC_2. Then from b, parallel to eC_2, draw a line to intercept the perpendicular diameter A_2S in the point C_3. (This at once shows bad to be an octant.) In like manner in each of the successive stages, the centre of description for the next circle of duplicate area may be found.

Plate II.



twice the half-octantal segment Be; and Be is twice the octantal segment BH; that is to say the half-octantal segment $\beta a \delta$ is equal to the two half-octantal segments in the octant $b a d$, to the four half-octantal segments in the quadrant BA_2D, and to the eight half-octantal segments in the semicircle BAD. The same reasoning applies to the quarter-octantal segment, described with the next radius and centre of description C_5, which will equal the sixteen quarter-octantal segments contained in the semicircle BAD; and so on to the ultimate limit, when the semicircle having been divided into the greatest possible number of equal arcs, further bisection (or division) becomes impossible.

The novelty in the ultimate result arrived at by this process, consists in obtaining a definite aliquot cyclal segment *similar* to the last or ultimate segment of the semicircle, resulting from repeated bisection carried to any definite stage, and *equal in area* to the sum of the areas of all those last or ultimate similar segments contained in the semicircle. Now if triangles be inscribed in all of the (similar) segments—by drawing the chords of the similar arcs—from the outermost to the innermost, it is manifest that the resulting similar triangles will be areally proportional to each other in the same ratios as the segments in which they are inscribed; and since the relative values of the successive radii are known, the values of the successive triangles may be readily computed.

Now, if the trigonometrical computation be made it will confirm what has just been shown by geometry, viz., that each of the *lunar** segments or triangles, which has the diameter, or a part of the diameter, of the primary circle for its chord or its base, has the same ratio and relation to the radius of the greater circle to which it belongs as the similar segment or triangle in the semicircle has to the radius of the primary circle. The relative areal ratios of the successive greater circles (of successive duplicate area) to the primary are as follows:—

The first	is twice	the primary.		The seventh	is	128 times	the primary.
„ second	„ 4 times	„	„	„ eighth	„	256 times	„ „
„ third	„ 8 times	„	„	„ ninth	„	512 times	„ „
„ fourth	„ 16 times	„	„	„ tenth	„	1,024 times	„ „
„ fifth	„ 32 times	„	„	„ eleventh	„	2,048 times	„ „
„ sixth	„ 64 times	„	„	„ twelfth	„	4,096 times	„ „

At the twelfth stage of the process, then, the greater circle, to which the *lunar* segment

* It is to be understood that the expression *lunar* is here used to distinguish the segment which has the diameter (or a part thereof) of the primary circle for its chord, and the arc of the greater circle as its curvilineal boundary.

last described belongs, is 4,096 times as great in area as the primary circle; and at the same stage it will be found that the number of similar segments in the semicircle amount in number to 4,096: and, as we have shown, the *one* corresponding *lunar* segment will then equal in area the sum of the areas of the 4,096 lesser similar segments. But, since the one *lunar* segment belongs to a circle 4,096 times areally greater, its ratio and areal relation to its own circle is precisely the same as that of each one of the lesser similar segments to *its* own circle.

The modern mathematician succeeds, by a process of successive bisections, in measuring one section of an inscribed reg. polygon of 4096 sides and the corresponding one section of a circumscribed reg. polygon of the same number of sides, and finds the difference between them exceedingly small, and to be diminishing in such a manner that if the process were supposed carried somewhat further it would seem to disappear altogether, and, as the corresponding section of the circle is evidently greater than the one and less than the other, he concludes that three things—(1) a complete areal circle, (2) an inscribed, and (3) a circumscribed, reg. polygon—which he knows with the utmost certainty are unequal, are, at the same time, equal each to each; a conclusion which, if Euclid had not been dead for so very long a time, would be enough, one might suppose, to make his bones rattle in their coffin. Let us see whether the precise nature and locality of the fallacy cannot be illustrated and made clear by the above *(lunar)* method of presenting the case.

The modern mathematician having measured the one side or section of the inscribed reg. polygon of 4,096 sides argues in this manner: "If I multiply the one side of the inscribed reg. polygon by 4,096, I shall get a complete lineal reg. polygon of 4,096 sides, and, if I multiply the area of the triangle—of which the one side of the polygon is the base and the centre of the circle the vertex-point of the vertical angle—by 4,096, I shall get the complete areal polygon. Just in like manner, if I multiply the small arc—corresponding to the one side of the reg. polygon—by 4,096, I shall get the complete lineal circle, and, if I multiply the sectional figure—bounded by the small arc and the two radii which join its extremities with the centre of the circle—by 4,096, I shall get the complete areal circle. Now, as far as the ratio is concerned, I may just as well take the one side and the small arc, or the sectional triangle and corresponding sector of the circle, as take the entire polygon and complete circle, for, since both are multiplied by the same quantity (4,096), the ratio is of course unaltered." Herein is what may be perhaps more correctly termed an unsound inference than an unsound assumption, but which, whatever it may be correctly called, undoubtedly falsifies the conclusion based upon it. In the first place, he does not in fact get either the complete circle or the entire polygon by multiplying the sections, but he gets the *same sections* on an enlarged scale. Is he quite aware of this?—that is to say, has he clearly

apprehended the circumstance? Perhaps he has; and he may say: "Well, never mind about the *forms* of the geometrical figures, the products represent respectively lineal or areal quantities equal to the lineal or areal polygon and to the lineal or areal circle." But, unmindful, if not unaware of the circumstance just referred to, he has, in the process of getting the measurement of the one side or section of the reg. polygon (by repeated bisection), multiplied his result at each stage, and, deeming it of no importance, has taken no note of the fact.* Consequently his measurements of the small sections expressed as ratios refer to the enlarged circle—4,096 times (lineally or areally as the case may be) as large as the primary—and are expressed in terms of the diameter or radius of that large circle. In order, therefore, to apply his measurements to the ratios of the entire polygon and circle, he must find a means of reducing the *scale* of his greatly enlarged standard circle to that of the primary, at the same time that, from the small section, he builds up the complete geometrical figure. He cannot do this by division, for then he would simply reduce the section of the large circle to a similar section of the primary.

Let us see what has to be done. Look again at the figure (Plate 11). The *lunar* segment, on the diameter of the primary, at the 12th stage, equals in area the sum of the areas of the 4,096 segments in the semi-circle of the primary. Is it not, then, manifest, that the *differences* of all the sections or small areal segments between the perimeter of the half-polygon (of 4,096 sides) and the semicircle's circumference, collectively expressed as a single *difference* or area *in terms of the same circle*, must be 4,096 times as great as the *differential* area of the one small segment between the one small side of the polygon and the corresponding fractional arc? The area of the *lunar* figure contained by the quadrantal arc and the two radii of the second circle, equals the area of the semicircle of the first. The area of the *lunar* figure contained by the octantal arc and the two radii of the third circle, also equals the area of the semicircle of the first. The area of the *lunar* figure contained by the half-octantal arc and the two radii of the fourth circle, equals that of the primary semicircle and that of each of the preceding figures; and so on, however far the process may be carried. Accordingly, at the 12th stage, the area of the figure bounded by the two radii and the small arc (the 4096th sectional part of a semicircle) equals the area of the entire primary

* To the question, "Why does he multiply, or why has he multiplied his results at each stage, as you say, that he has?"—the answer is:—Because he wishes to commence his numerical statement of the measurement (which is his final result) with a digit either preceding or following the decimal point. If he did not so multiply his results he would have to follow the decimal point with a formidable array of oughts before he could put down the first *live* figure to express his measurement of the extremely minute sector or side of the polygon, as the case may be, in terms of the primary areal circle or of its radius.

semicircle.* Evidently, therefore, we are now dealing with a section which is the 4,096th part of a circle 4,096 times as large as the primary circle, and when we now take the same numerical figures, which express the areal value of the semicircle, to indicate, also *as a ratio*, the area of the enlarged section which *absolutely* equals the area of the entire semicircle, those figures, in the case of the section, refer to the great circle to which the section belongs. It is quite true that the area of the segment, bounded by the arc and its chord, which equals the collective areas of the 4,096 segments contained between the circumference of the semicircle and its 4,096 chords, has simply the same ratio to its circle as each of the lesser similar segments has to the primary, but when the enlarged segment is referred to the primary circle instead of to its own great circle, *its* ratio becomes increased 4,096 times.

Let us illustrate this by its application to the case of the circle's true area. That area's ratio to the square on the diameter as four, is, by our geometrical demonstration 3·1426968... and the area of the greatest or *ultimate* inscribed reg. polygon (which has hitherto been mistaken by mathematicians for that of the circle itself), is 3·14159265... showing the ratio of the difference to be ·0011042... What, then, taking these data, will be the ratios of the areas of the one-sided section of an inscribed reg. polygon of 4,096 sides and of the corresponding fractional section of the circle, when the area of the square on the diameter of that circle = 4 × 4096 ? First, ·0011042... divided by 4,096 = ·0000002695... and, adding this to the ratio of the complete ultimate inscribed reg. polygon, 3·14159265... + ·0000002695...=3·141592923... Now, this will evidently be the true ratio with regard to the 4096th fractional section of the circle; but since an inscribed reg. polygon of 4096 sides is obviously less than the *ultimate* inscribed reg. polygon (which is an inscribed reg. polygon of the greatest possible number of sides), the actual difference must be somewhat greater than ·0000002695... By Legendre's computation the one-sided section of the circumscribed reg. polygon of

* The usual method of circumscribing as well as of inscribing reg. polygons may be applied to the successive *lunar* arcs. To each su ccessive arc there will be two or four sides of the circumscribing polygon corresponding to the two or four sides of the inscribed polygon. When the octagonal half polygon is circumscribed about the semicircle a quadrilateral section of a polygon is circumscribed about the enlarged quadrant. When the polygon about the semicircle becomes sixteen-sided the quadrilateral polygon is about the enlarged octant: when the polygon about the semicircle becomes thirty-two-sided the quadrilateral is about the half-octant: and so on, as shown in Plate 11. In each case the area of the enlarged section of the circumscribed polygon, bounded by its four sides and the two rādii connecting it with the centre of the enlarged circle to which it belongs, equals the area of the entire half polygon circumscribed about the primary semicircle. Finally there will be a *lunar* double *ultimate* sector of the great circle, equal in area to the primary semicircle, a two-sided section of the circumscribing and a two-sided section of the inscribed polygon, equal in area respectively to half of the entire circumscribing and half of the entire inscribed polygon.

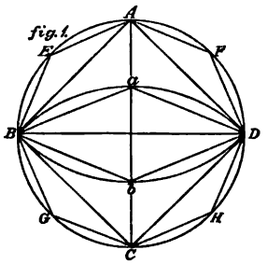
4,096 sides=3·1415933... and of the inscribed reg. polygon=3·1415914... Deducting the latter from the former, the difference=·0000019 ... is about 7 times as great as the difference shown above between the corresponding sections of the *ultimate* inscribed reg. polygon and the circle. Therefore, when it is considered that the inscribed reg. polygon of 4,096 sides must be very sensibly *less* than the *ultimate* inscribed reg. polygon*, the circumscribed polygon of 4,096 sides sensibly *greater* than the *ult.* circumscribed polygon, and that the *ult.* circumscribed reg. polygon is *greater* than the circle about which it is circumscribed, it will be readily apprehended that Legendre's computation (which is in itself quite reliable), when correctly applied, strictly harmonizes with and supports our geometrical demonstration of the circle's ratio—which equals 3·1426968...

The demonstrations to some problems of considerable interest may be derived from the method of the *lunar* analysis. The following may serve as examples:—

Problem 1.

To describe in a square a rhombus such that the areal spaces on both sides of the rhombus, between the rhombus and a regular octagon circumscribed about the square, shall together equal the area of the square. (Fig. 1.) *See also Plate* 12.

Let *A B C D* be the given square, and *A E B G C H D F* the reg. octagon circumscribed about it. Draw the diameter *B D* of the octagon, and with its point of bisection as centre and half the diameter as radius, describe a circle circumscribing the octagon; of which circle the sides of the given square are the quadrantal chords. With centre *C* and the quadrantal chord *C B* as radius, and with centre *A* and chord *A B* as radius, respectively, describe the quadrantal arcs *BaD*, *BbD*, of which diameter *B D*, of the first circle, is the chord common to both. To *a* and to *b*, respectively from *B* and *D*, draw the four octantal chords *Ba*, *Da*, *Bb*, *Db*. The rectilineal figure *BaDb*, bounded by the four octantal chords, is the rhombus required, the spaces on each side of it (which together with the area of the rhombus itself constitute the entire area of the octagon) together equalling in area the given square.



Because the radius of each of the quadrantal arcs *BaD*, *BbD*, is equal to the diagonal of a square of which the radius of the circle, in which the octagon is inscribed, is one of the sides, each of the two quadrantal segments *BaD*, *BbD*, of which the

* The actual difference, as appears above, is about ·0000012...

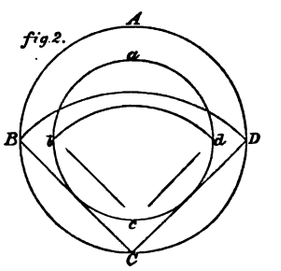
diameter is the common chord, is areally twice as great as each of the four quadrantal segments *B E A*, *A F D*, *D H C*, *C G B*, of the circle; and therefore the two lunes (or curvilineal figures bounded by the semicircles externally and by the quadrantal arcs on the diameter internally) equal the two inscribed triangles *B A D* and *B C D* which together constitute the given square. Now, since the greater and lesser quadrantal segments and the greater and lesser octantal segments respectively are similar figures, the lesser octantal segments have the same ratio to the greater which the lesser quadrantal segments have to the greater, and therefore the four greater octantal segments, which bound the rhombus, equal the eight lesser octantal segments which bound the octagon. Hence each of the spaces, on the two sides of the rhombus, bounded externally by the four chords of the lesser octantal segments and internally by the two chords of the greater octantal segments, equals each of the two triangles which together constitute the given square, and therefore these two spaces (or rectilineal lunes), taken together are equal to the given square.

Corollary.—Hence, manifestly, the square and the rhombus together equal the octagon. Also, since (by the Demonstration of Hippocrates, page 115), the two lunes are together equal to the inscribed square, each of the rectilineal figures between the rhombus and the octagon equals each of the lunes.

Problem 2.

To describe a quadrantal sector equal in area to a given circle. (Fig. 2.) *See also Plate* 12.

Let *a b c d* be the given circle. Draw the diameter *b d* and the chords *c d* and *c b*.* With centre *c* and radius *c b* describe the quadrantal arc *b d*. Bisect the diameter in the centre of the circle, and with that centre and radius equal to *c b* describe the greater circle *B A D C*. Draw the quadrantal chords *C D* and *C B* of the greater circle, and with centre *C* and radius *C B* describe the quadrantal chord *B D*. The figure bounded by the quadrantal arc *B D* and the right angle *B C D* is the required figure equal in area to the given circle.



Because the square on the chord *b c* of the lesser circle has twice the area of the square on the radius (Prop. 25, *P.F.*) the quadrantal segment *b d*, on the diameter, equals in area the two quadrantal segments *b c* and *d c*. Therefore the figure *b d c* bounded by the quadrantal arc *b d* and the angle

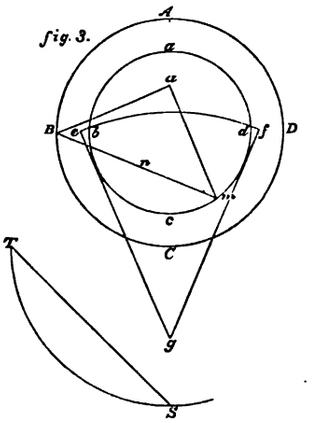
* The quadrantal chords *c b* and *c d*, and the diameter *b d* of the lesser circle, have been omitted in the engraving, but will be found in the enlarged figure of Plate XII.

b c d, equals in area the semi-circle *b a d*. Now the quadrantal chord *c b* of the lesser is the radius of the greater circle, and therefore, by Prop. 25, *P.F.*, that circle has twice the area of the lesser circle. But therefore, also, the greater figure *B D C*, bounded by the quadrantal arc and the right angle, has twice the area of the lesser similar figure *b d c*. Now, the lesser figure has been shown to equal half of the circle, *a b c d*. Wherefore the greater figure, *B D C*, bounded by the quadrantal arc and the right angle, equals in area the given circle *a b c d*.

PROBLEM 3.

To describe an octantal sector equal in area to a given circle. (Fig. 3.) *See also Plate* 12.

Let *a b c d* be the given circle. With the circle's centre and the quadrantal chord as radius, describe a second circle *A B C D*, and draw the diameter *B D* thereof. With centre *C* and quadrantal chord *C B* of the second circle describe a third circle *B D S T*. Bisect the quadrant *B D**, of the third circle in the point *a*, and draw the chord *B a*, of the octant. From *a*, at right angles to *B a*, draw *a m* equal to *B a*. Join *B m*, and bisect it in *n*. With the centre of the second circle and radius equal to *m n*, cut the diameter *B D*, of the second circle, in the points *e* and *f*. With centre *e* and radius equal to the quadrantal chord *S T* (or *B D*), of the third circle, cut the perpendicular diameter thereof in the point *g*; and with centre *g* and radius *g e* describe the arc *e f*. The figure *e f g* bounded by the arc *e f* and the angle *e g f*, shall be the required octantal sector.



Because the radius of the third circle is to the radius of the fourth circle† as the side to the diagonal of a square, and because the chord *B a* is proportional in the same ratio to the chord *e f*, and because the arc *B a*, of which *B a* is the chord, is an octant, the arc *e f* is an octant. Now it has been shown in the preceding proposition that the figure bounded by the quadrantal arc *B a d* of the third circle and the angle *B C D* formed by the two quadrantal chords *B C* and *D C* of the second circle, equals the first

* Because the quadrantal chords *B C* and *D C* are necessarily at right angles, and are the radii of the arc *B D*, that arc is a quadrant.

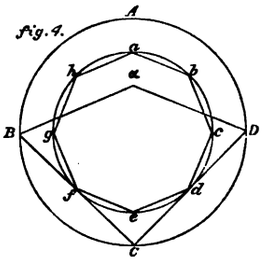
† The third circle is *B D S T*, and the fourth is that of which *g* is the centre and the arc *e f* an octant.

circle *a b c d*. But since the radius of the fourth is to the radius of the third circle as the diagonal to the side of a square, and it has been shown that the figure bounded by the arc *e f* and the angle *e g f* is an octantal sector of the fourth circle, the area of that figure must equal the area of the quadrantal sector *B a D C* belonging to the third circle. It has been shown that the quadrantal sector *B a D C* equals in area the circle *a b c d*; wherefore the octantal sector *e f g*, bounded by the arc *e f* and the angle of 45° *e g f* is equal to the given circle *a b c d*.

PROBLEM 4.

To describe the quadrantal section of a regular octagon, equal in area to a given regular octagon.

Let *a b c d e f g h* be the given reg. octagon. (Fig 4.) With the centre *o* of the octagon and the distance *o g* describe the circle circumscribing the octagon. With the same centre and radius equal to the quadrantal chord of the first circle describe a second circle *A B C D*. With centre *C* and the quadrantal chord *C B* of the second circle describe the quadrantal arc *B a D*. Bisect the arc *B a D* in the point *a* and draw the chords *B a* and *D a* of the two octants. The figure *B a D C*, bounded by the two octantal chords *B a* and *D a* and the two quadrantal chords *B C* and *D C*, shall be the figure required. Bisect the quadrants *B C* and *D C* in the points *m* and *n* respectively, and draw the chords of the octants.*



Then, because the radius of the quadrantal arc *B a D* is to the radius of the second circle as the diagonal of a square to its side, the quadrantal segment *B a D* equals the two quadrantal segments *B m C* and *D n C* together, and, for the same reason, the two octantal segments *B a* and *D a* equal the four octantal segments *B m*, *C m*, *C n*, *D n*, together. Therefore the triangle *B a D* equals the two triangles *B m C*, *D n C*; and it follows that the two triangles *B a D* and *B C D* together equal the half octagon *B m C n D*. Now the radius of the second circle is to the radius of the first as the diagonal to the side of a square. Hence the first circle has half the area of the second, and the reg. octagon, inscribed in the first, has half the area of a reg. octagon inscribed in the second. But the figure *B a D C*, bounded by the triangles *B a D* and *B C D*, has been shown to equal half of a reg. octagon inscribed in the second circle; wherefore that figure equals the whole of the given reg. octagon inscribed in the first circle.

* The letters *m* and *n* and the six octantal arcs are omitted in the above figure, but will be found in Plate XII.

Fig. 1.

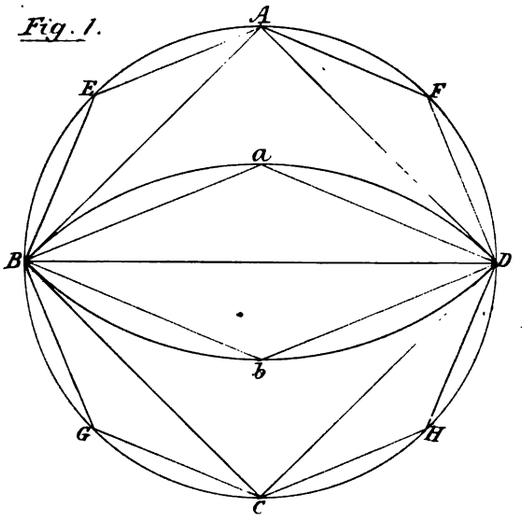


Fig. 2.

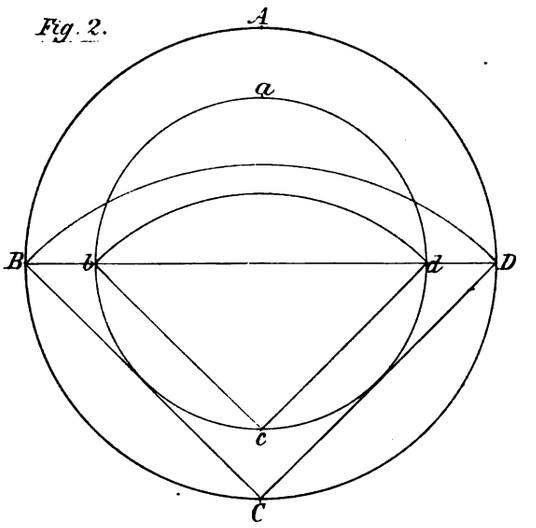


Fig. 4.

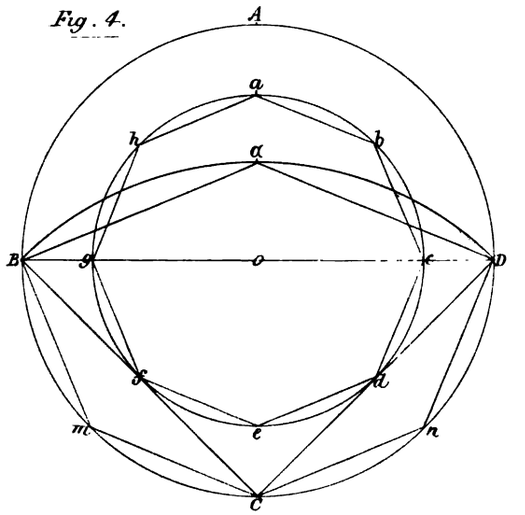


Fig. 3.

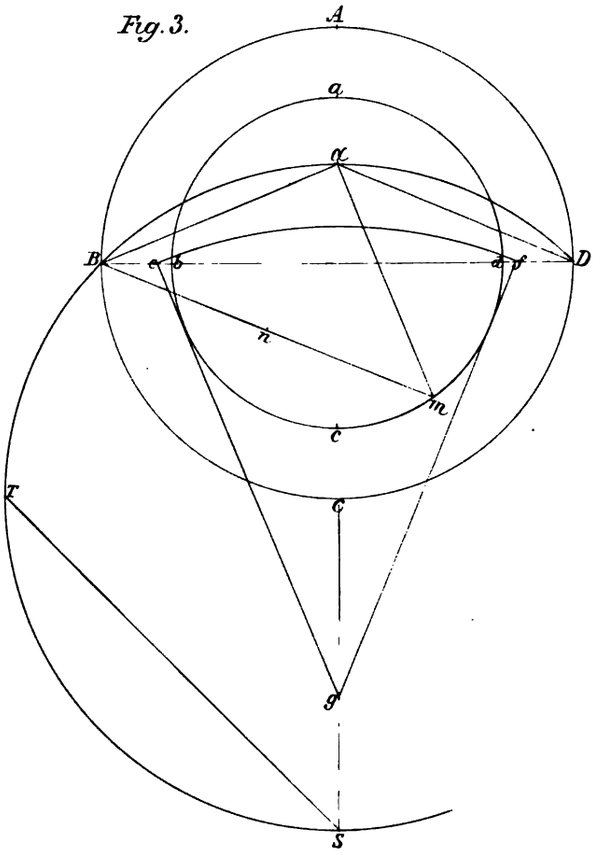
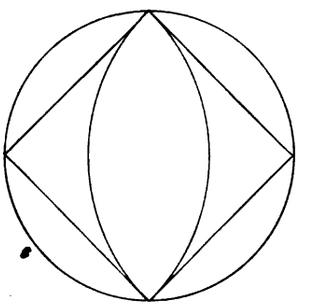


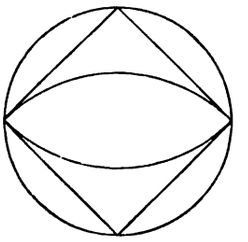
Fig. 5.



Scholium.—The angle $B\,C\,D$ which bounds the figure on one side is a right angle, for it is contained by the two quadrantal chords of the semicircle $B\,D\,C$; and the angle $B\,a\,D$, bounding the figure on the other side, is a right-angle and a-half (135°) for the two radii and the chord of each octant constitute an isosceles triangle of which the two angles at the base must together contain a right angle and a-half, and the angle $B\,a\,D$ manifestly equals the two equal angles at the base of each of the isosceles triangles.

Corollary.—Hence if an isosceles triangle of which the vertical angle equals a right angle and a-half, be described on the opposite side of the same base as a right-angled triangle, of which the vertical angle contains the right angle, and the half of a reg. octagon be circumscribed about the right-angled triangle, the areas of the two triangles together will equal the area of the half octagon. But if a half octagon be inscribed in the right-angled triangle, the areas of the two triangles together will then equal the area of the entire reg. octagon to which the inscribed half belongs.

Fig. 5 represents the Lune of Hippocrates, doubled. The areal correlations of the circle and the two figures within it are these: The difference in area of the inscribed square and the circle equals the area of the curvilineal figure bounded by the two quadrantal arcs; and the difference in area of that curvilineal figure and the circle equals the area of the inscribed square.*



* See Plate XII., repeating the figures belonging to these problems on an enlarged scale.

PART FOURTH

DUPLICATION OF THE CUBE

PART FOURTH.

DUPLICATION OF THE CUBE

(AND PROBLEMS CONNECTED THEREWITH)

DUPLICATION OF THE CUBE.

PROP I. PROBLEM.

To find two mean proportionals between two given straight lines.

Fig. 1. Let AF the greater and AB the lesser be the two given straight lines; it is required to find two mean proportionals between AB and AF.

Arrange the two lines in one line ABF (from which AB is cut off in the sectional point B). Take any convenient part of BF as an addition to AB such that when added to AB the line so obtained will be certainly less than the first of the two mean proportional s and, since the one-fifth part of BF added to AB will compound a lineal magnitude certainly less than the first of the two mean proportionals required, cut off in the point c one-fifth of BF. Find (by Prop. 19, *P.F.*) and cut off in the point d, a third proportional to the lines AB and Ac; so that $Ad : Ac :: Ac : AB$. Find (by the same Prop.), and cut off in the point e, a third proportional to the lines Ac and Ad; so that $Ae : Ad :: Ad : Ac$. From point e draw a line perpendicular to AF, and with centre e and distance eF cut the line so drawn in the point g. Join Bg. From point c, and parallel to Bg, draw cn intercepting eg in point n; and from point d, also parallel to Bg, draw dm intercepting eg in point m. From cF, as an addition to the line Ac, cut off a part cD, equal to the part gn in the line eg. From dF as an addition to the line Ad, cut off dE equal to the part gm in the line eg. The lines AD and AE shall be the two mean propor-

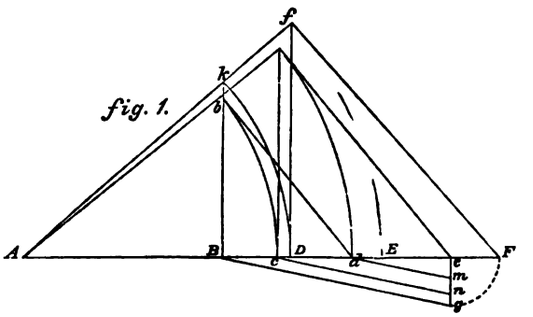


fig. 1.

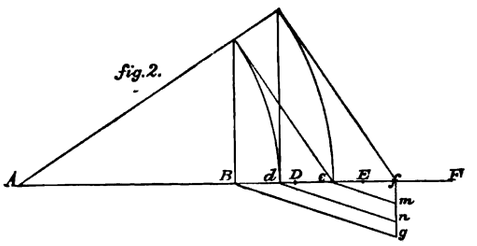
tionals between the two given straight lines; such that $AB : AD :: AD : AE :: AE : AF$.

Because eg equals eF, and the lines cn and dm are parallel to the line Bg, the line eg is divided similarly to the line Be, and the ratios of the proportionals gn gm, ge, are the same as the ratios of the proportionals Bc, Bd, Be. Hence when the part cD, equal to gn, is added to Ac, the part dE, equal to gm, added to Ad, and eF equal to the whole of ge, added to Ae, the increased lines are also in continual proportion, namely, $AB : AD :: AD : AE :: AE : AF$. Wherefore AD and AE are the two mean proportionals between the lines AB and AF.*

PROP. II.—PROBLEM.

To find the dimensions of a cube of which the solid contents shall be twice that of a given cube.

Fig. 2. Let the line AB equal a lineal side of the given cube. Produce the line AB through B, and make BF, the produced part, equal to AB, so that the whole line AF equals twice AB. Cut off in the points D and E two mean proportionals between the lines AB and AF (Prop. I.), so that $AB : AD :: AD : AE :: AE : AF$. Then must the line AD equal a lineal side of a cube having twice the solid contents of the given cube.



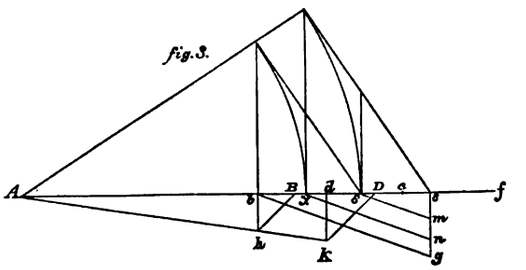
For AD is by the preceding proposition, the second lineal magnitude of four continual proportionals of which AF, the last, is twice as great as AB, the first, and by Euclid's XI. 33, "*Similar parallelopipeds are one to another in the triplicate ratio of their homologous sides.*" Now AB is to AF in the triplicate ratio of AB to AD (Euclid's V., *Def.* 11), and therefore AD is a lineal side of the required cube, homologous to the lineal side AB of the given cube; for the ratio of the required cube to the given cube is two to one, and the ratio of the line AF to the line AB is two to one. Wherefore the required cube is that of which a lineal side is equal to the line AD.

* The demonstration, if found unobjectionable, is sufficient. It may, however, be independently tested and verified, as shown in the figure : viz., from B and D draw perpendiculars, and from D and E, with radii AD and AE, describe arcs meeting the perpendiculars in points k and f. From f at right angles to AF, draw fF. Then by similarity of the triangles the ratios are shown.

PROP. III. PROBLEM.

To find the dimensions of a cube of which the solid contents shall be one-half those of a given cube.

Fig. 3. Let $A\,D$ equal one of the lineal sides of the given cube. From $D\,A$ cut off any convenient part, such that the remainder is certainly less than a lineal side of the required cube, and, since one-third of $D\,A$ will be manifestly sufficient, cut off in point b one-third thereof. Produce $A\,D$ through D and in the produced line take the point f such that $A\,f$ equals twice $A\,b$. Find (by Prop. I.) and cut off in points d and e two mean proportionals between $A\,b$ and $A\,f$.* From point d, at right angles to $A\,d$, draw a line $d\,k$ of any convenient length, and join $A\,k$ and $D\,k$. From point b, at right angles to $A\,b$, draw $b\,h$ intercepting $A\,k$ in the point h. From h, parallel to $k\,D$, draw $h\,B$ intercepting $A\,D$ in B. The line $A\,B$ shall be a lineal side of the required cube.



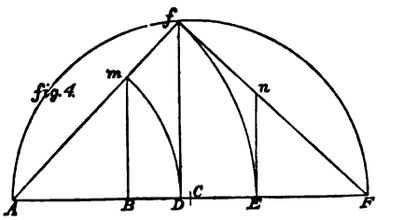
Because the triangles $A\,b\,h$ and $A\,d\,k$, and also the triangles $A\,h\,B$ and $A\,k\,D$, are similar, the line $A\,D : A\,d :: A\,B : A\,b$. But $A\,D$ equals a lineal side of the given cube, and $A\,d$ is the first of two mean proportionals between $A\,b$ and $A\,f$. Therefore the line $A\,D$ is the second of four continual proportionals of which $A\,B$ is the first and of which the last must equal twice $A\,B$. Now the ratio of the given cube to the required cube is 2 to 1, and by the preceding proposition, when the fourth lineal magnitude of four continual proportionals equals twice the first, the second equals a lineal side of a similar parallelopiped equal to twice the parallelopiped of which the first is a lineal side. Wherefore the line $A\,B$ equals a lineal side of the required cube.

* So that $Ab : Ad :: Ad : Ae :: Ae : Af$.—We have here taken, as in Prop. I., the one-fifth part ($b\gamma$) of bf to obtain the lineal magnitude less than the first of the two mean proportionals between $A\,b$ and $A\,f$. Then, as in Prop. I., $b\gamma$, by an addition equal to $g\,n$, becomes $b\,d$; $b\,\delta$, by an addition equal to $g\,m$, becomes $b\,e$; $b\,\epsilon$, by an addition equal to $g\,\epsilon$, becomes $b\,f$. The upper lines in the figure verify the first series of proportionals.

PROP. IV. PROBLEM.

To find a cube which shall have the ratio to a given cube of 3 *to* 1.

Fig. 4. Let the line $A\,B$ equal a lineal side of the given cube. Produce $A\,B$ through B, and make $A\,F$ greater than $A\,B$ in the ratio of three to one. Find, and cut off in points D and E, two mean proportionals between $A\,B$ and $A\,F$. (Prop. I.) Then, because "*similar parallelopipeds are one to another in the triplicate ratio of their homologous sides*" (Euclid XI. 33), and the triplicate ratio of $A\,D$ to AB is 3:1, the line $A\,D$ equals a lineal side of the required cube.

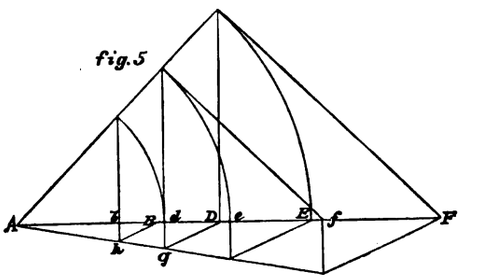


Corollary.—It follows that, in the like manner—*viz.*, by increasing the line, equal to a side of the given cube to the same ratio required for the second cube to have to the first cube, and finding two mean proportionals between the first and the increased line—a second cube may be found greater than a given cube in any ratio required; for, as shown, the first of the two mean proportionals will equal a lineal side of the required cube.

PROP. V. PROBLEM.

To find a cube which shall have the ratio to a given cube of 1 *to* 3.

Fig. 5. Let the line $A\,D$ equal a lineal side of the given cube. Cut off from $D\,A$ a convenient part, such that the remainder will be less than a lineal side of the required cube, and, since one-half of $D\,A$ will be certainly less than the required side, cut off in point b one-half of $A\,D$; so that $A\,b$ is to $A\,D$ as 1 to 2. Produce $A\,D$ through D, and make the production $A\,f$ proportional to $A\,b$ in the ratio of 3 to 1. Find (by Prop. 1), and cut off in points d and e two mean proportionals between $A\,b$ and $A\,f$. From point d, perpendicular to $A\,d$, draw $d\,g$ of any convenient length, and join $A\,g$ and $D\,g$. From point b, perpendicular to $A\,b$, draw a line intercepting the line $A\,g$ in point h, and from point h, parallel to $g\,D$, draw a line intercepting $A\,D$ in point B. Then, because $A\,f$ is to $A\,b$ as 3 to 1, and $A\,d$ is the first of two mean propor-



tionals between $A\,b$ and $A\,f$, and, by the similarity of the triangles, $A\,B$ is to $A\,b$ as $A\,D$ is to $A\,d$, the line $A\,B$ must be the first of four continual proportionals, of which $A\,D$ is the second, and of which the fourth must have the ratio to the first of 3 to 1. That is, the triplicate ratio of $A\,B$ to $A\,D$ is 1 to 3, and therefore (by Euclid's XI. 33) $A\,B$ is a lineal side of the required cube, having the ratio to the given cube of one to three.

Corollary.—It follows that, in the like manner,—*viz.*, by cutting off from the line equal to a side of the given cube a part sufficient to make the *remaining* line certainly less than a side of the required cube, producing the line so that the *produced* line shall have that ratio to the *remaining* line which the second cube is required to have to the first, then finding two mean proportionals between the *remaining* and the *produced* line, and, lastly, obtaining a line having the same ratio to the *remaining* line which the side of the given cube has to the first of the two mean proportionals—the lineal side of a second cube, less than a given cube in any required ratio, may be found; for, as shown, the line last obtained must equal a side of the required cube.

SCHOLIUM.—In this, and also in some of the preceding propositions, the *modus operandi* with regard to the construction may be considerably varied, in the following manner, without essentially altering or deviating from the principle on which it is based:—(Fig. 5 *b*). Instead of cutting off a part such that the remainder will be certainly less than the required side, cut off a convenient part such that the remainder will be certainly greater than the required side; and since three-fourths of $D\,A$ will be certainly greater than the required side, cut off in point b one-fourth of $D\,A$, so that $A\,b$ is to $A\,D$ as 3 to 4. As before, make the production $A\,f$ (of $A\,D$) proportional to $A\,b$ as 3 to 1; and find in the points d and e two mean proportionals between $A\,b$ and $A\,f$ (by Prop. I.). Then from point D, perpendicular to $A\,D$, draw $D\,n$, of any convenient length, and join $A\,n$ and $d\,n$. From point b, parallel to $d\,n$, draw $b\,m$ intercepting line $A\,n$ in point m; and from point m, perpendicular to $A\,f$, draw $m\,B$ intercepting line $A\,f$ in point B. It follows, as before, that $A\,B$ must be a lineal side of the required cube.

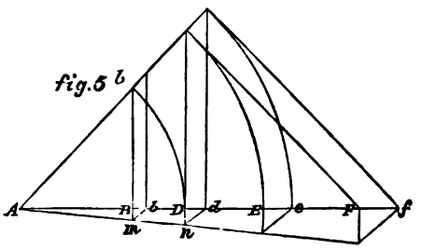


fig. 5 *b*

(In the diagram the upper lines verify the demonstration by shewing that in each of the two lines $A\,f$ and $A\,F$, the intermediate points, between b and f and B and F respectively are the mean proportionals required.)

The arc to *d* has been omitted by the Engraver.

PROP. VI. PROBLEM.

To find a cube equal in solid contents to a given sphere.

Inscribe a square in the great circle of the sphere (*i. e.*, the circle of a section of the sphere through its centre), and also circumscribe a square about that circle. Increase a side of the inscribed square by one-ninth of itself, and then find a mean proportional between that increased side and a side of the circumscribed square.* By quadrature of circle (Prop. IV., Part Fifth), the mean proportional so found is the side of a square having the same area as the circle, and, therefore, that mean proportional equals also a lineal side of a cube having the same solid contents as the given sphere.

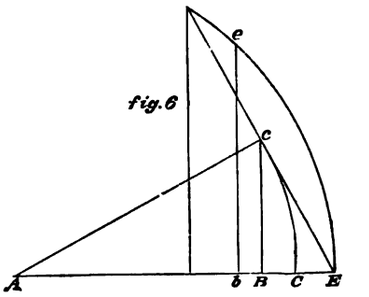


fig. 6

Fig. 6. Let $A\,b$ be a side of the square inscribed in the great circle of the given sphere and $A\,B$ the increased line obtained by adding to $A\,b$ the one-ninth of itself. From point b, perp. to $A\,E$, draw $b\,e$ equal to $b\,A$. Then $A\,E$ equals the diagonal ($A\,e$) of the square on $A\,b$. And (by Prop. 14, Part First) $A\,C$ is the mean proportional between $A\,B$ and $A\,E$. Wherefore $A\,C$ is the side of the required cube equal in solid contents to the given sphere.

PROP. VII. PROBLEM.

To find a Sphere equal in solid contents to a given Cube.

From a line equal to a lineal side of the cube take a convenient part such that the *remainder* is certainly less than a side of a square inscribed in the great circle† of the required sphere. Increase the lineal *remainder* by one-ninth of itself and then find a mean proportional between that increased line and the diagonal of a square on the lineal *remainder*. Find a fourth proportional having the same ratio to the side of the given cube as the diagonal of the square has to the mean proportional. Then the fourth proportional equals the diameter of the great circle of the required sphere.

Fig. 7:—Let $A\,D$ be equal to a lineal side of the given cube; and from $D\,A$ take the part $D\,b$, equal to one-third of $D\,A$, leaving the remainder ($A\,b$) certainly less

* Or thus:—Describe a square having the diameter of the sphere's great circle for its diagonal. Increase a side of that square by one-ninth of itself, and find a mean proportional between that increased side and the diameter of the sphere's great circle.

† The circle of the sphere's section through its centre.

than a side of a square inscribed in the great circle of the required sphere. Increase $A\,b$ by $b\,B$, equal to one-ninth of $A\,b$. Take, in the line $A\,D$, $A\,E$ equal to $A\,e$ the diagonal of a square on the lineal remainder $A\,b$, and find the mean proportional $A\,C$, by Prop. 18, Part First, between $A\,B$ and $A\,E$ Then, because the diagonal of an inscribed square equals the diameter of the circumscribing circle, $A\,E$ equals the diameter of the circle circumscribing the square of which the lineal remainder $A\,b$ is a side, and, since, by Prop. 20, Part First, $A\,F : A\,D :: A\,E : A\,C$,* the line (by Prop. IV., Part Fifth) $A\,F$ equals the diameter of the great circle of the required sphere.

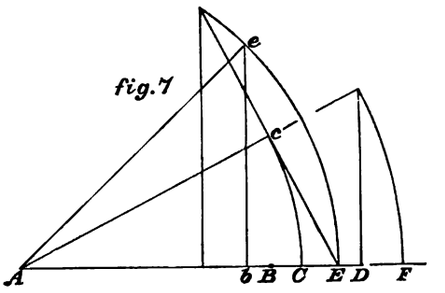


fig.7

PROP. VIII. PROBLEM.

To find a Sphere having twice the solid contents of a given Sphere (i. e., to duplicate a Sphere).

(1.) Find a cube equal in solid contents to the given sphere (Prop. VI.). Duplicate that cube (Prop. I.); and find a sphere equal to the duplicated cube (Prop. VII.). The sphere so found will be the required sphere.

(2.) Fig. 8:—Let the line $A\,B$ equal the diameter of the great circle of the given sphere. Produce $A\,B$ through B, and make $A\,F$ equal to twice $A\,B$. Find, by Prop. I., two mean proportionals $A\,D$ and $A\,E$ between $A\,B$ and $A\,F$. Then $A\,D$, the first of those two mean proportionals, must equal the diameter of the great circle of the required sphere.†

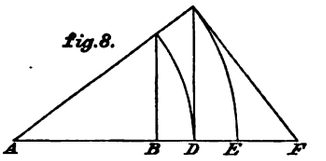


fig.8.

AUGMENTATION AND DIMINUTION OF THE SPHERE.—It is evident that in a similar manner to the foregoing. and by aid of the preceding propositions on the augmentation and diminution of the cube, a sphere greater or less, in any required ratio, than a given sphere may be readily found.

* The engraver has omitted the perpendicular from point *C*.

† See also Fig. 2 belonging to Prop. 2. We have shown in Part First (Prop. 38) that areal circles are one to another as the circumscribed (or inscribed) areal squares. Consequently it follows that this problem (Prop. VIII.) is virtually a repetition of Prop. 2 (*To duplicate a given cube*), merely substituting the diameter of the great circle of the given sphere for the side of the given cube.

PART FIFTH

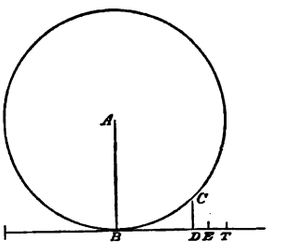
QUADRATURE OF THE CIRCLE

PART FIFTH.

QUADRATURE OF THE CIRCLE.

THE RATIO OF THE OCTANTAL ARC TO ITS SINE.

Rolling a Circle or Arc on a Straight Line.—Def.: Let B and C be any two points at a distance less than or not exceeding a quadrantal arc, in the circumference of a circle of which the radius is $A\,B$; and let B be the point at which the tangent line $B\,T$ touches the circle. When the circle (or arc) is ideally so moved that each successive point in the circumference from B to C touches successively the line $B\,T$, in such wise that each successive point in the arc touches a successive point in the straight line, until finally the terminal point C of the arc touches a point in the straight line, the circle (or arc) is said to be rolled upon the straight line from B to C of the arc. From the terminal point C, in the first position of the arc, draw a perpendicular $C\,D$ to intercept the tangent line, then $B\,D$ on the tangent line equals the sine of the arc, and (hypothetically) assuming E to be the point at which the terminal point C touches the tangent line, when the arc is rolled from B to C, the distance $D\,E$, which measures, on the straight line, the part by which the arc exceeds the sine, is called the *difference*, and may also be called the *advance* of the arc.



The unknown point on the tangent line, at which the terminal point is (hypothetically) assumed to arrive, is said to be *indicated* by the point to which the letter of reference is attached.*

[Instead of an *indicated* point, a definite point of arrival on the tangent line may

* It is not thereby primarily affirmed that the point marked by the *indicating* letter is the actual point at which the terminal point of the arc touches the tangent line, but merely that it may be the actual point and otherwise is near to the actual point of arrival. The *indicated* point is to be considered tentative and indefinite until justified and made definite by the reasoning, which in the meantime must apply only to the lineal magnitude, terminated by the indicated point, of which the existence is evident and manifest but of which the quantity is uncertain until determined and made certain by the demonstration itself.

be taken, and a terminal point for the arc be *indicated*, which it is then hypothetically assumed will arrive at the definite point on the tangent line; *e.g.*, let E on the tangent line be the definite point, and C on the circumference be the indicated point: the circle (or arc) is then said to be rolled from B to E of the tangent line.]

POSTULATE.—Let it be granted that a circle or any arc of a circle may be rolled upon a straight line tangent to the circle.

PROP. I. THEOREM.

The part by which an octantal arc is greater than its sine is an aliquot part of the arc.

FIG. 1. With centre A and radius $A\,B$ describe the octantal arc $B\,M$, and draw the tangent $B\,T$. From point M, perpendicular to the tangent, draw $M\,D$ intercepting the tangent in point D. Let the arc be rolled upon the tangent line $B\,T$, until the extremity M of the arc touches the line, and let O *indicate* the point at which M arrives. Cut off from radius $A\,B$ in the point K the one-n^{th} part (so that $K\,B$ is to $A\,B$ as n-1 to n),* and with centre K and radius $K\,B$ describe the second octantal arc $B\,n$. From n draw a perpendicular intercepting the tangent line in point C. Draw the sine $N\,M$, of the first arc $B\,M$, and the sine $n\,p$, of the second arc $B\,n$. Then, because the first octantal arc ($B\,M$) is to the second ($B\,n$) as n to n-1, the *advance* or *diff.* $D\,O$ of the first is to $C\,d$ that of the second as n to n-1. That is to say, the *diff.* of the first exceeds that of the second by one-n^{th} of itself.

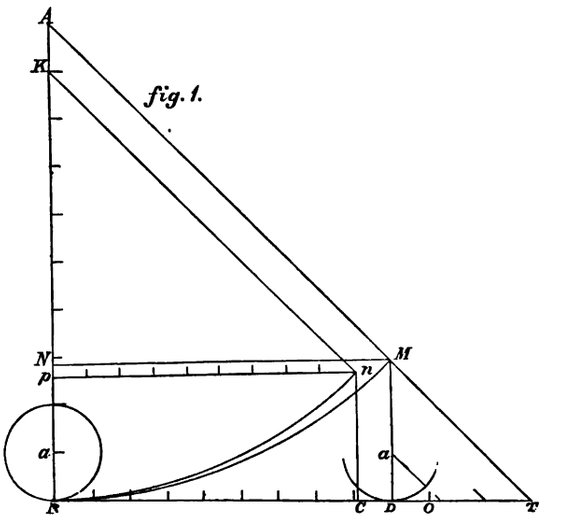


fig. 1.

* In the diagram the divisions of the radius and arc lengths are into *tenths*. In this there is no assumption: ten stands for n, one for one-n^{th}, nine for $n-1$ parts. Of course, in the diagram *some* specific divisional standard must be adopted. As will be seen, further demonstration establishes certainty that ten is the *natural* unital standard of the octantal radius, and therefore it is much preferable to adopt ten at once as the representative of n. If, however, the reader please, he may instead take 8, 9, 11, 12 or some other integral quantity, and the reasoning with regard to this primary theorem will still hold good.

By the construction, the ratio of BC to BD must be that of $n-1$ to n. Divide the sine NM (or BD) of the first octant into $n-1$ equal parts, and divide the sine $p\,n$ (equal to BC) of the second octant into $n-1$ equal parts; and suppose the *diff.* (Cd) belonging to the second octant also divided into $n-1$ equal parts. Let point d be the actual point* in the line BT at which the terminal n of the second octant arrives, and let O be the actual point at which the terminal M of the first octant arrives. Then, by the construction, the ratio of Cd to DO must be that of $n-1$ to n. Also, by the construction, dO (the difference between the arc-lengths of the two octants) must be one-n^{th} of BO the arc-length of the greater, and dO must include one-n^{th} of the sine-length of the greater octant, that is, must include the equivalent of CD. It follows that Cd must equal CD, because dO is necessarily greater than CD by the one-n^{th} of *diff.* which belongs to each one-n^{th} of the sine of the greater octant.† Now make DO one-n^{th} of BO; then the ratio of CD to DO is that of $n-1$ to n, and all the requisite conditions are fulfilled, which cannot be so by supposing Cd unequal to CD. Therefore the points d and D are coincident and the same; the point D is the actual point in the line BT at which the terminal n of the second octant must arrive; and the part DO, the *diff.* by which the octantal arc BM is greater than its sine, is an aliquot part (namely, the one-n^{th}) of its arc-length.

Again (Same Fig. repeated):—With centre a and radius aB, the one-n^{th} of radius AB, describe the small circle BUV touching the tangent line BT at point B. Let the same small circle be supposed also upon the tangent line at D, that is touching the line BT at the point D. Then DC must equal the sine of the small octant, with the perpendicular radius aD, towards the extremity B of the tangent line; for the radius of the small circle is one-n^{th} the radius AB, and DC is one-n^{th} the sine BD of the first octant BM. Now if the small circle BUV be rolled, from its first position at B, through n successive octants, the terminal point of the last octant must touch the line BT at the point indicated by O, because O is the *indicated* point at which M the terminal point of the octant BM arrives, and each octant of the

* The two letters D and d refer in fact to the same point, namely, that in which the perpendicular MD intercepts the tangent line. This (the use of two letters) is done to prevent the appearance of an unwarranted assumption, for we must not *assume* that the terminal point n of the second octant arrives exactly at D, and therefore the actual but unknown point at which it must arrive is *indicated* by the letter d. The mode of demonstration, here primarily adopted, is dependent upon proof that the points d and D are identical. In the figure the letter d is omitted, to avoid complexity.

† In other words, dO must include and be compounded by the equivalent of CD together with one-n^{th} of DO; DO must include and be compounded by the equivalent of Cd together with ne-n^{th} of DO; dO and DO must each be to Cd in the ratio of n to n—1.

small circle is one-n^{th} of the octant $B\,M$. Hence, when the small circle has rolled through n-1 octants it must touch the tangent line at the point D, for the entire distance will be then compounded of $n-1$ sines, each equal to $C\,D$ (and collectively equal to $B\,C$) and of $n-1$ *diffs.*, each equal to one-n^{th} of $D\,O$. With the addition of another octantal arc-length, that is if the small circle be rolled through one more (the n^{th}) octant, the entire distance becomes $B\,O$, and the last arc-length (belonging to the small circle), compounded of a sine-length equal to $C\,D$ and a one n^{th} of $D\,O$, must have the ratio to the $n-1$ *diffs.* (belonging to the $n-1$ sines which together equal $B\,C$) of n to $n-1$. Consequently $D\,O$ equals the whole arc-length of one small octant, namely the sine and *diff.* thereof together. Wherefore the *difference* $D\,O$ of the given octant $B\,M$ is shown to equal the one-n^{th} of its arc-length.*

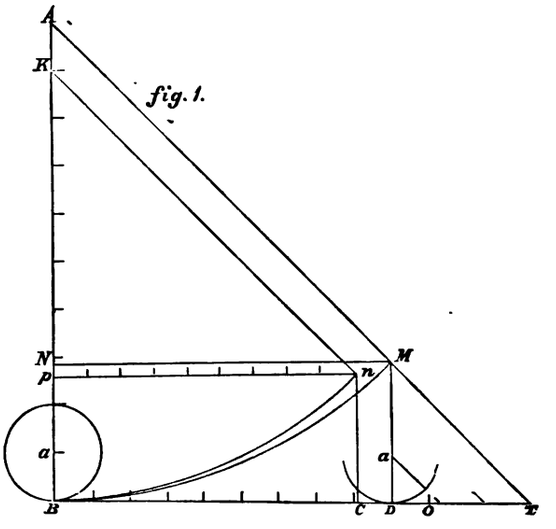


fig. 1.

" SCHOLIUM.—Since the magnitudinal difference between the sine and arc of the octant has now been shown to be the one-n^{th} of the arc, and since the *diff.* itself may be represented by a lesser octant one-n^{th} the magnitude of the greater (and therefore equal to that *diff.*), it becomes evident that the arc-length may be expressed in terms of the sine as follows:— Putting A for the *arc*, S for the *sine*, and d for the *difference*. . . . . Then

* It is unnecessary to point out to the experienced mathematician that, as soon as the validity of this theorem is admitted, the time-honoured value assigned to π by orthodox mathematics becomes manifestly untenable. The younger student and the general reader, however, may be glad to have this plainly shown. The value assigned as the ratio of the circumference to the diameter of the circle is 3·14159 . . . and accordingly the ratio of the octantal arc-length to the radius as unity would be ·785398. Now the ratio of the octantal sine, to the radius as unit, is, beyond the possibility of dispute or doubt ·7071068 . . . (*i.e.* the sq.-root of one-half). Deducting the sine from the supposed arc-length (785398—7071068) the remainder or *diff.* (·078291) is, manifestly, *not* an aliquot divisor of the sine or of the supposed arc-length.

$A = S + d = S + \frac{S}{n} + \frac{d}{n} = S + \frac{S}{n} + \frac{S}{n^2} + \frac{d}{n^2} = S + \frac{S}{n} + \frac{S}{n^2} + \frac{S}{n^3} + \frac{d}{n^3} = S + \frac{S}{n} + \frac{S}{n^2} + \frac{S}{n^3} + \frac{S}{n^4} + \frac{d}{n^4}$ = and so on; finally $A = \frac{n}{n\text{-}1} S$.

It cannot be stated as a *direct* geometrical corollary, but it is *virtually* a corollary, and can easily be shown to be a mathematical corollary, to the foregoing demonstration, that: If the arc-length of an octant be divided into *ten* equal parts, the *diff.* is magnitudinally equal to and *in fact* constitutes one of those *ten* equal parts." (Quoted from "Demonstration of the Circle's Ratio.")

PROP. II. THEOREM.

The part by which the octantal arc is greater-than its sine is the tenth part of the arc-length.

There are three distinct modes by which mathematical certainty as to the soundness of this theorem may be established, namely, (1) Direct geometrical experiment: (2) Trigonometrical demonstration: (3) Inductive reasoning based on known facts.

(1) DIRECT EXPERIMENT.—One method of applying this is to take the diagram on a moderately large scale, and with a pair of compasses test the ratio by measurement; such a method carefully conducted will readily make apparent that the *diff.* is considerably greater than one-eleventh and considerably less than one-ninth of the arc-length. We are prepared to hear it said "Oh! this is a mechanical and not a geometrical mode of procedure. By methods of this description mathematical certainty cannot be established." But with such a view of the matter we cannot agree.—It is geometrical. By our Prop. 4, P.F., and Prop. 6, Part Second, the arc and the sine may be divided into any required number of equal parts. By dividing the arc into a great number of equal parts and taking the chord-lengths collectively, the length of the arc and of its *diff.*, may be approximately obtained as straight lines. By dividing the approximate arc-length into ninths, tenths, elevenths, and so on, we may then make geometrical comparison with the approximate *diff.*-length, basing the comparison on the axiom that a greater and a lesser magnitude are unequal. If the experiment be fairly and correctly made, the process will be sound and mathematical, and the conclusion be quite certain. We cannot primarily in such a manner directly determine with mathematical exactitude or certainty the actual ratio. For such a purpose it would be quite unsuited and geometrically inadmissible. But such is not

the present purpose, for it is no longer requisite. The preceding theorem demonstrates the *diff.* to be an aliquot divisor of the octantal arc-length. Therefore if it be manifest, or can be made manifest in the geometrical sense, that the ratio is greater than one-eleventh and less than one-ninth, it follows necessarily that the actual ratio is one to ten.

Notwithstanding the horror with which such a *machine* is likely to be regarded by devotees of (what they suppose to be) pure mathematics, we have arranged a little contrivance in metal, which we have ventured to call the Mechanico-Mathematical Instrument, primarily for the purpose of illustrating the ratio as shown by the geometrical process of rolling an arc, but which also serves very well to roughly demonstrate the present theorem, for it is constructed, as may be readily ascertained by examination, on the ratio of one to ten, and, since it works quite correctly and smoothly, the fact is thereby made manifest that the actual ratio must be that of one to ten, for if it were not very nearly one to ten the instrument could not work, and, if *nearly* one to ten, it must, by the preceding theorem, be *exactly* one to ten.

(2.) TRIGONOMETRICAL DEMONSTRATION.—We have shown* that the process of trigonometrically comparing the magnitude of reg. polygons, the perimeters of which are alternately inscribed in and circumscribed about the circle, does not result as hitherto supposed in furnishing the ratio of the circle's circumference. Nevertheless the process itself is mathematically sound and quite reliable as furnishing a much nearer approximation than is requisite for the purposes of this theorem. The ratio which it furnishes (making the *diff.* a unit) is 1 to 10·03178 . . . As we have shown, this result correctly applies to the perimeter of the inscribed *ultimate* reg. polygon, and, since the inscribed polygon is necessarily less than the circle which circumscribes it, the approximative demonstration of trigonometry (with regard to the octantal arc and its *diff.*) is *really* that the true ratio must be exactly or very nearly indeed that of 1 to 10. We succeeded some time since,† by a process called "The Radial Duplication of the Bisected Arc" in showing this same trigonometrical result in a manner more direct and simple than that of inscribing and circumscribing polygons, and which may be said to approximate very nearly to a geometrical demonstration in the more direct sense, for by it a part of the tangent line is finally cut off demonstrably equal to the ultimate sine when enlarged to the scale of the primary arc itself: That is to say,—the primary sine length (to the octantal arc) being ·7071068 . . . the ultimate sine length, cut off on the tangent line, is shown to be ·785398 . . . furnishing the ratio, as stated above, of 1 to 10·03178 . . .

* "Lecture First on the Circle and Straight Line." See also "Obs. on the Lunar Process," page 118.

† "Lecture First on the Circle and Straight Line."

(3.) INDUCTIVE REASONING BASED ON KNOWN FACTS.—Having already carefully stated the argument* which we wish to present under this head we will here repeat that previous statement:—

"(*a.*) The radius of the octant is a *natural unit* or complete numerical quantity and is necessarily divisible into *unital increments*, each of them numerically complete in itself, and in quantitative composition similar to the primal unit of which it is an unital increment; therefore the radius of the octant is naturally divisible into ten equal parts, each of which is naturally divisible into ten equal parts; and so on: that is to say, the radius of the octant is *decimally* divisible, and can be *only decimally* divided, into unital increments, for the primary unital increment is the one-tenth part, the secondary unital increment is the one-hundreth part, the tertiary unital increment the one-thousandth part, and so on.

(*b.*) Because the radius of the octant is naturally divisible into ten equal parts, the arc and the sine of the octant are each of them also *naturally* divisible into ten equal parts.

(*c.*) Because the *sine* and the *diff.* together compound the arc-length of the octant and because each unital increment of arc contains a primary unital increment of *sine* together with a primary unital increment of *diff.*, which primary unital of *diff.* is compounded of a secondary unital increment of *sine* together with a secondary unital increment of *diff.*, which secondary unital increment of *diff.* is compounded of a tertiary unital increment of *sine*, together with a tertiary unital increment of *diff.*, and so on,—therefore when the arc of the octant is evenly divided into ten parts, the magnitudinal *difference* between the arc and the sine is equal to one of those ten equal parts. *Wherefore*:—The ratio of the octantal arc to its sine is the ratio of *ten* to *nine*."

SCHOLIUM.—It has been hitherto, we believe, generally, if not universally, assumed that the denary Scale of Number, commonly called the Arabic system of Notation, is a highly ingenious artificial arrangement devised by the human mind for the purposes of computation. We feel sure, however, that attentive consideration of the foregoing demonstration and exposition will result in such assumption being relinquished as no longer tenable. Because it becomes evident, in the first place, that magnitude is *fundamentally* identical in plan with quantity or number; that is to say, both are grounded on the same unital basis and denary scale; for the "number" or "quantity" *ten* is a unital increment complete in itself and compounded of ten lesser *unitals* each of which possesses a like completeness. For example, in the application of quantity to comparative measures of length, the foot and the yard are artificial units:

* "Demonstration of the Circle's Ratio," published in 1879.

the foot might be made to contain thirteen or fourteen inches instead of twelve, and the yard four or five feet instead of three. But the properties and relations of the number *ten* cannot be conferred upon the number *nine* or upon any lesser compound of complete units. There may be *ten* ones and *nine* hundreds: the ten ones constitute a complete number or quantity divisible into ten similar and equal units; but the nine hundred are only a part of ten hundred which is the complete *unital* (or denary unit) to which they belong. This may be plainly seen by taking *nine* ones nine times, when the product (81) is evidently a number or quantity *dissimilar* to and having essentially different relations from those of the nine; whereas *ten* ones taken ten times constitute simply a greater *unital*, similar to and possessing relations essentially the same as the lesser. In the same case is the octantal radius. We may call it *one* or *ten* or *a hundred*; but, whether we call it so or not, it is a *natural* unital and cannot be made otherwise. We may try, for instance, to call it *nine*; by so doing we render it incomplete and reduce it to a part (nine-tenths) of a whole octantal radius. If we call it *eight*, it is, as such, reduced to eight-tenths, and so on, It becomes, therefore, also evident, secondly, that the system of complete (denary) unital increments, compounded of complete decimal units and decimally divisible, both with regard to "number" and "magnitude," is not an artificial human device, but is Natural, in, at least, the human sense, and if Artificial at all can only be so in the absolute *supernatural* sense, that is, as having been devised by the Creator, and utilised by Him, as the magnitudinal and numerical basis, in His fundamental plan of that general arrangement—the parts of which, collectively, as a whole, we call NATURE.

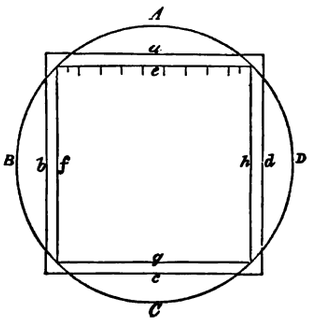
GEOMETRICAL QUADRATURE OF THE CIRCLE.

The ratio of the circle's circumference to its diameter, as determined by the foregoing demonstration, may be expressed as $\frac{10}{9} \times \sqrt{8} = 3{\cdot}14268\ldots$, when the diameter equals 1. The direct demonstration, however, is that the octantal arc $= \frac{10}{9} \times \sqrt{{\cdot}5}$ when radius equals 1; and, since the octant's sine then $= \sqrt{{\cdot}5}$, it follows that the perimeter of the inscribed square is to the circle's circumference as nine to ten.

QUADRATURE.—This ratio being mathematically established the actual geometrical quadrature becomes a very simple matter.

PROP. III. PROBLEM.

To describe a Square lineally equal to a given Circle (or vice versâ).



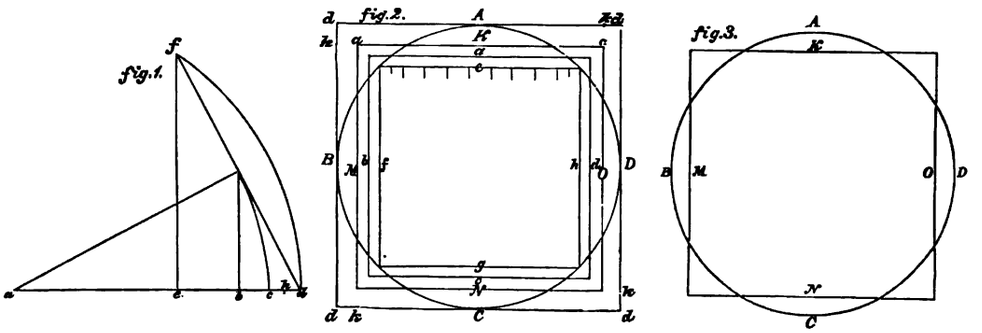
Let the square *e f g h* be inscribed in the circle *A B C D*. Divide one of the sides (as *e*) into nine equal parts. Bisect one of the nine equal parts and increase the length of each of the four sides of the square one-ninth of itself by addition of half the one-ninth part to each extremity (of each of the sides). The perimeter of the greater square *a b c d*, so constructed, equals, by the foregoing geometrical demonstration, the circumference *A B C D* of the circle: for the lineal circle and the greater square both have the same ratio to the perimeter of the inscribed square, namely, that of ten to nine, and therefore they are equal.

RECTIFICATION.—It has been shown and is mathematically well established that the area of the circle is equal to the circle's circumference multiplied by half the radius, or, what comes to the same thing, to half the circumference multiplied by the radius. We are not aware that it has hitherto been shown, but, if not, it may readily be shown, that the side of a square areally equal to a circle must be a mean proportional between a straight line which equals a fourth part of that circle's circumference and the side of the circumscribed square. For, let radius equal one, then the side of the circumscribed square (or the diameter) equals two, and the fourth part of the circumference multiplied by two equals half the circumference multiplied by one which equals the circle's area. Put *a* for circle's area, *c* for circumference. Then obviously, $\frac{c}{4} : \sqrt{a} : : \sqrt{a} : 2$; because $\frac{c}{2} \times 1 = a$. For the *quadrature* above, we have shown how to describe a square of which each side equals a fourth part of the circumference of a given circle, and it is now made apparent that, to describe a square *areally* equal to a given circle, it is requisite to find a mean proportional between a side of the lineal square, which equals the circumference of the circle, and a side of the circumscribed square. The mean proportional will be the side of a square areally equal to the given circle.

PROP. IV. PROBLEM.

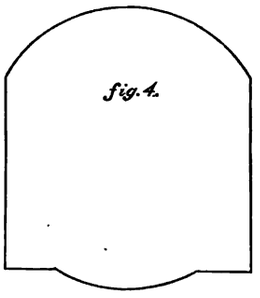
To describe a Square areally equal to a given Circle (or vice versa).

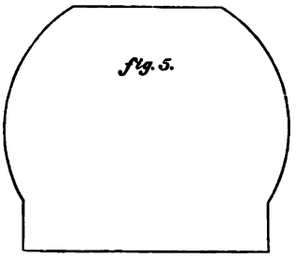
Figs. 1 and 2.—By Prop. III., taking *a d* the greater line (Fig. 1) equal to one side *A* (Fig. 2) of the circumscribed square *A B C D* (or to the diameter of the circle),



and *a b* the lesser line equal to one side *a* of the square *a b c d*, which *lineally* equals the circle's circumference,* we obtain the line *a c* as the mean proportional. Bisect *c d* in *k* (Fig. 1), and from each extremity of each side of the circumscribed square *A B C D*, cut off a part equal to *k d*. Hence we obtain the square *K M N O*, (Figs. 2 and 3) of which each side equals the (mean proportional) line *a c*; and which square therefore areally equals the given circle.

COROLLARY.—Since the square *K M N O* (Fig. 3) is areally equal to the circle *A B C D*, it follows that each of the four angular figures cut off from the square, at the corners, by the circle's circumference, areally equals each of the four segments cut off from the circle by the sides of the square; and, by the same demonstration, the compounded figures—Figs. 4 and 5—are each areally equal to a complete circle, of the circumference of which the curvilineal part of the boundary in the figure is a part, and also each equal to a square of which the base line in Fig. 5 is one of the sides.



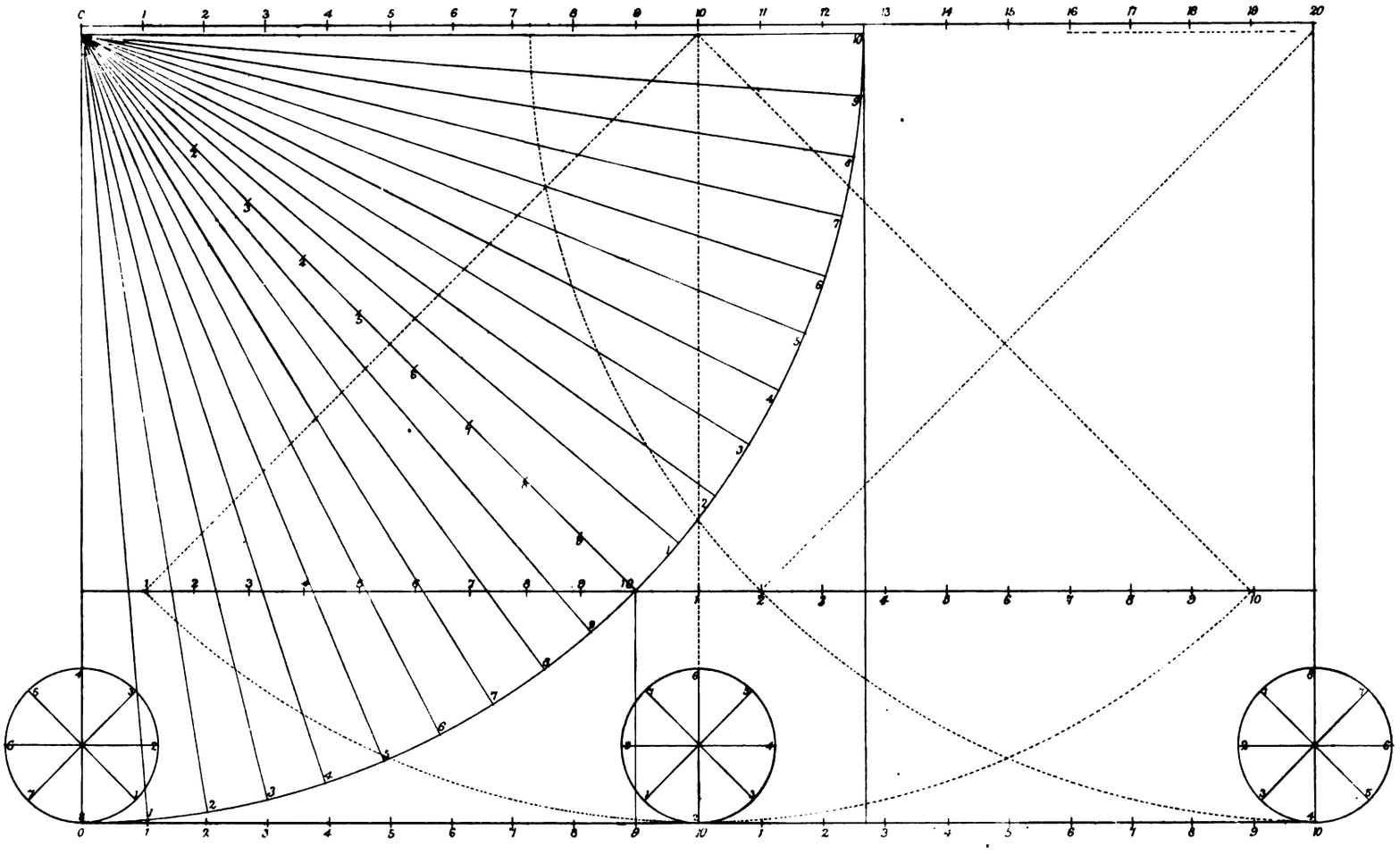


* Obtained by increasing each side of the inscribed square by one-ninth of itself, as shown for the Quadrature.

APPENDIX

TO

PART FIFTH.



The Mechanico-Mathematical Instrument.

Illustrating the Ratio of the Sine to the Arc of the Octant.

(2)

To Accompany the Mechanico-Mathematical Instrument.

GEOMETRICAL DEMONSTRATION

OF THE RATIO OF THE

CIRCLE'S CIRCUMFERENCE TO THE DIAMETER.

By JOHN HARRIS.

(Quoted, for the most part, from Mr. Harris' Second Lecture on "The Circle and Straight Line.")

WERTHEIMER, LEA AND CO., CIRCUS PLACE, FINSBURY CIRCUS.

LONDON, FEBRUARY, 1879.

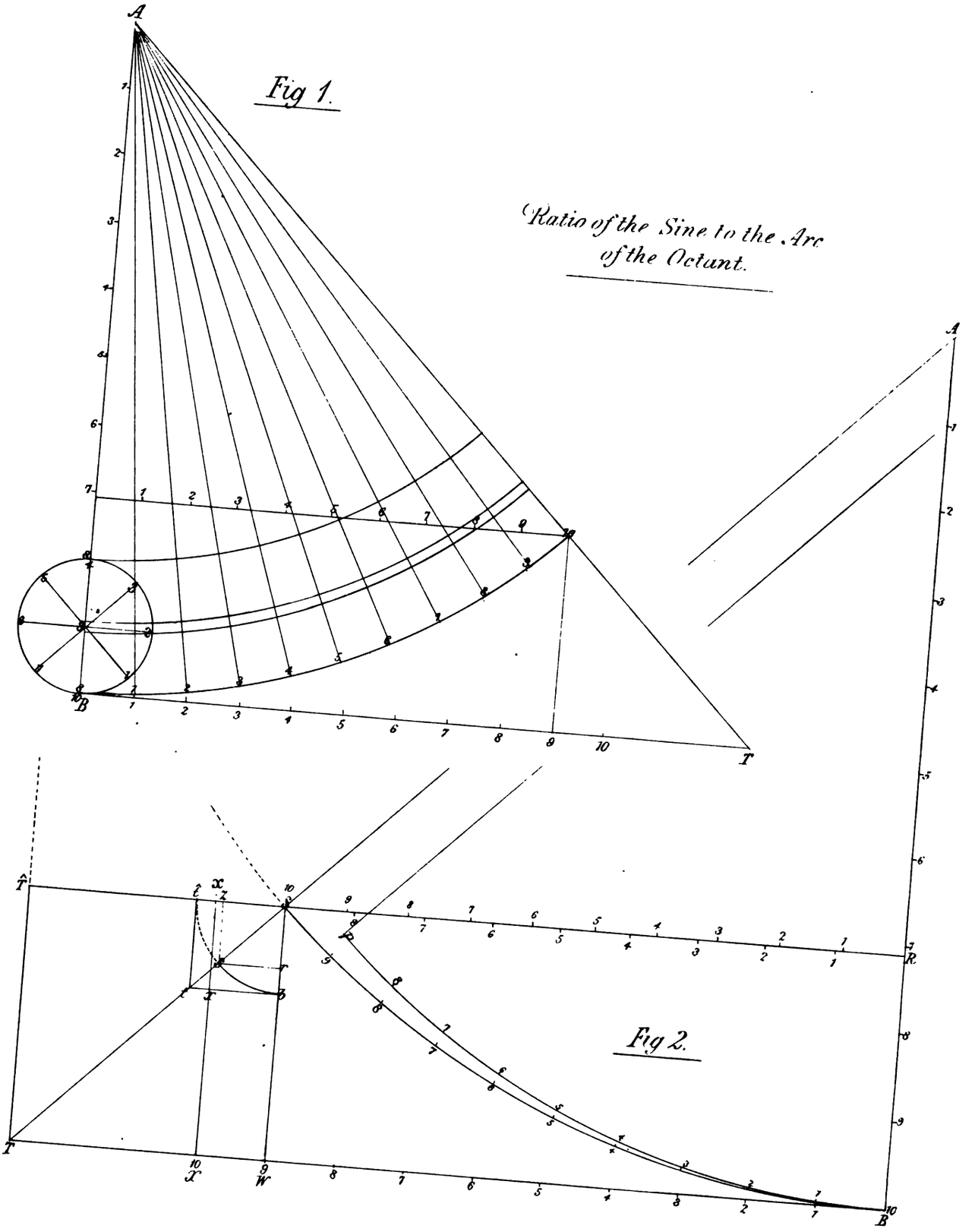


Fig 1.

Ratio of the Sine to the Arc of the Octant.

Fig 2.

GEOMETRICAL DEMONSTRATION

TO ACCOMPANY

THE MECHANICO-MATHEMATICAL INSTRUMENT

Illustrating the Ratio of the Circle's Circumference to the Diameter.

Explanatory of the Instrument.—(Fig. 1.) The radius of the small circular disc in front, is one-tenth the radius of the large octant. The *arc* of the large octant is divided into ten equal parts. Its *sine*-length is divided into nine equal parts on the (lower) flat side of the scale which represents the sine, and into ten equal parts on the (upper) bevelled side thereof. The tangent plate, upon which the octant and the small circle stand, is divided into ten equal parts between the two points intercepted by the two extremities of the octant respectively when, the radius at the one extremity being at first perpendicular, the octant is rolled upon the plate until the radius at the opposite extremity becomes perpendicular. The space on the plate between the two extreme points, so divided into ten equal parts, is called the arc-length of the octant.

Since the radius of the small circle is one-tenth the radius of the large octant, when the small circle is rolled from the same starting point as the large octant and is rolled through one complete revolution and a quarter of a second revolution—*i.e.*, through ten octants—the terminal extremity of the tenth lesser octant will arrive at the same point on the base line intercepted by the terminal extremity of the large octant when rolled through its own length: that is to say, the small circle in rolling one revolution and a quarter will roll through the arc-length of the greater octant; because one-tenth of that arc-length is the arc-length of each one of the lesser octants.

Definition.—If a perpendicular be drawn from the terminal point at the elevated extremity of the octant to the base line when the other extremity of the octant is upon the base line, and the octant be then rolled until (the elevated extremity becoming depressed) that terminal point be upon the base line, the space on the base line, between the point intercepted by the perpendicular so drawn and the point intercepted by the terminal point of the octant, is called *the advance* of the octant. The same space is also called *the difference* of the octant, because it is the measure of the longitudinal difference between the sine and the arc-length of the octant.

Demonstration that the arc-length of the octant may be divided into n equal parts, such that *the difference* (between the sine and arc-length) will be one of those n equal parts; and that, therefore, *the diff.* is an aliquot divisor of the arc-length and also of the sine-length. *Diagram, Fig.* 2.

(1.) Let the radius, AB, of the large octant, be divided into n equal parts.

Then, if with a lesser radius equal to one of those equal parts, an octant be described, the magnitude of that lesser octant will be one-*n*th the magnitude of the greater octant; and the sine of the lesser will be one-*n*th of the sine of the greater. Again: if with a lesser radius equal to two of the *n* equal parts of the primary radius *A B*, an octant be described, the magnitude of that lesser octant will be two-*n*ths the magnitude of the greater octant; and so on. Finally: if with a lesser radius equal to *n*-1 parts of *A B*, an octant be described, that lesser octant will be of one-*n*th less magnitude than the greater: and the sine of the lesser will be of one-*n*th less magnitude than the sine of the greater octant.

Now, since the octants are all similar arcs of the circle, the *difference* between the arc and sine in each octant is proportional to the similar *diff.* in each of the other octants respectively, in the same ratio as the ratio of the octant itself to each of the other octants respectively. Therefore the *diff.* in that lesser octant of which the magnitude is one-*n*th of the magnitude of the primary octant, is one-*n*th the *diff.* in the primary octant. And the magnitude of the *diff.* in that lesser octant which is of one-*n*th less magnitude than the primary octant, is one-*n*th less than the magnitude of the *diff.* in the primary octant; and so on.

Consequently, if *n* lesser octants be described each with a radius equal to one-*n*th of the primary radius *A B*, the *diff.* in each will be one-*n*th the *diff.* belonging to the primary octant; and the sum of the *differences* in all the *n* lesser octants taken together will equal the *diff.* belonging to the primary octant. The *diff.* belonging to the primary octant is therefore evenly divisible into *n* equal parts, each of which parts represents the *diff.* belonging to one of the *n* lesser octants.

Consequently, if the one-*n*th part be taken from the sine of the greater octant, the other *n*-1 parts (as one component of arc-length) represent a corresponding *diff.*, (as the other component of arc-length) containing *n*-1 parts of the *diff.* belonging to the complete greater octant: and the remaining one-*n*th part of the sine of the greater octant *similarly represents* a corresponding one-*n*th part of the *diff.* belonging to the greater octant.

Therefore *n*-1 parts of the greater *diff.* equals one of the *n* equal divisions of the greater sine: for, let the *diff.* itself be considered as a lesser octant, then the one-*n*th part of the *diff.* thereof belongs equally *either* to the other *n*-1 parts, as its sine, *or* to the one-*n*th part of the greater sine; and, since the greater *diff.* equals the lesser octant, the lesser *diff.* is to the greater *diff.* as the lesser sine is to the greater sine, and as the greater *diff.* is to the greater octant.

Wherefore: it is shown *that* an octant may be evenly divided into *n* equal parts such that the *difference* between the arc-length and sine of that octant is one of those equal parts: and *that*, the sine of the octant being also divided into the same number

of equal parts, the sine of a lesser octant, of which the arc-length equals the *diff.* belonging to the greater octant, is equal to one of those equal parts of the sine of the greater octant. *Corollary.*—Hence, because the *diff.* is one-*n*th the arc-length of the octant, if the sine of an octant be evenly divided into *n*-1 equal parts, the *diff.* equals one of those equal parts.

(2.) The same demonstration may be presented somewhat differently as follows:—Since the *radius* of the octant is the integral standard of relative lineal magnitude, *it* (the radius) may be evenly divided into *n* equal parts. And, since each of those equal divisional parts of the radius represents a lesser octant with its proportional sine, the *octant* itself and its *sine* may be (each) similarly divided into *n* equal parts. Therefore the *diff.* between the octant and its sine may also be correspondingly divided into *n* equal parts. With a lesser radius, equal to one-*n*th of the primary radius, let a lesser octant equal to one-*n*th of the primary octant be described, and let that lesser octant be evenly divided into *n* equal parts. Evidently the *diff.* between that lesser octant and its sine is one-*n*th the similar *diff.* in the primary octant. Because the *diff.* is evenly divisible into *n* equal parts corresponding to *n* equal divisional parts of the sine, each one-*n*th part of the sine represents a corresponding one-*n*th part of the *diff.*, together with which it compounds one-*n*th part of the octant's arc-length. But the whole *diff.* is a part of the octant, and may be itself considered as a lesser octant* in which the *diff.* must have the same ratio to *its* sine as the (whole) greater *diff.* has to the sine of the primary octant. The sine of that lesser octant must therefore be equal to one of the *n* equal parts of the sine of the primary octant: for each of the *n* parts of the primary sine has a one-*n*th part of the *diff.* belonging to it, as co-component of arc-length. Let the one-*n*th part of the lesser octant be the *diff.* belonging to that lesser octant, and the remaining *n*-1 parts be the sine; and let that lesser sine equal one of the *n* equal parts of the greater sine. Then the ratio of the one part to all and to each of the other parts respectively will be that ratio imperatively required, for the lesser *diff.* will then be one-*n*th the greater *diff.*; the lesser sine will be one-*n*th the greater sine; the lesser sine will be *n*-1 parts of the *n* equal parts of the lesser octant; the greater sine will be *n*-1 parts of the *n* equal parts of the greater octant. And if the lesser octant be rolled upon its base line from the point *S* at the terminal extremity of the greater sine, and the greater octant be rolled upon its base line from the point *B* at the original extremity of that octant, the terminal extremity of each

* Since the *diff.* is evenly divisible into *n* equal parts, it (the *diff.*) evidently has *some* definite magnitudinal ratio to the octant's arc length. Whatever be that ratio, if with a lesser radius having *that same ratio* to the primary radius which the *diff.* has to the octant's arc-length, a lesser octant be described, that lesser octant must magnitudinally equal the *diff.* belonging to the primary octant.

octant will arrive at a point in its base line, intercepted by the same perpendicular.* Now if any other magnitudinal ratio be assigned to the lesser octant which equals and represents the *diff.* of the primary octant, those parts will no longer be correlated in the manner required. Therefore the sine of the lesser octant is one-nth the sine of the primary octant; and the magnitudinal ratio of the *diff.* between the sine and arc-length of the octant to the arc-length, is shown to be the ratio of one to n.

Again: Let the *diff.* in the primary octant be considered as before the arc-length of a lesser octant; and let the sine and the *diff.*, belonging to the greater octant, be each divided into n equal parts. Now, from the greater sine, at the end thereof adjoining the *diff.*, let one of the n equal parts be removed. Evidently the n-1 *corresponding* parts of the *diff.*, which belong as co-components of arc-length to the remaining n-1 parts of the sine, will (collectively) equal the one-nth part of the sine: for the one-nth part of the *diff.* which remains, requires for its co-component of arc-length, a *corresponding* sine equal to one-nth part of the greater sine, and, consequently, the n-1 parts of the *diff.* must equal the one-nth part of the greater sine. *Wherefore:* if the sine of an octant be evenly divided into n-1 equal parts, the *difference* between the sine and arc-length of that octant is equal to one of those equal divisional parts of the sine.

(3.) The demonstration may be also varied in the following manner:—Let the radius $A\,B$ of the primary octant be divided into n equal parts, and, taking a centre at one-nth less distance than A from B, and a radius equal to n-1 parts of $A\,B$ the primary radius, describe therewith a second octant $B\,p$ of one-nth less magnitude than the primary octant. Now if the second octant $B\,p$ be rolled until its extremity be upon the base line $B\,T$, the point of contact upon that line at which p arrives, must be the point W at the extremity of the sine-length of the primary octant. For since the second octant is equal to n-1 parts of the primary octant, n-1 parts of the primary *diff.* belong to that second octant as the *diff.* thereof. But, evidently, the other nth part of the *diff.* requires, as its co-component of arc-length, a sine-length, *corresponding* to it, having the same ratio to *it* which the n-1 parts of the primary sine have to the n-1 part of the primary difference. *Wherefore:* The *diff.* in an octant contains one-nth of the sine and one-nth of the *diff.* belonging to that octant; and if the arc-length of an octant be evenly divided into n equal parts, the *diff.* equals one of those equal divisional parts.

* Or if both octants, the greater and the lesser, be rolled upon the same base line $B\,T$, the lesser from the point W, and the greater from the point B, the terminal extremities of both will *necessarily* arrive at the same point x in that base line: because the arc-length of the lesser equals the *diff.* of the greater octant.

Scholium.—Since the magnitudinal difference between the sine and arc of the octant has been now shown to be the one-*n*th of the arc, and since the *diff.* itself may be represented by a lesser octant one-*n*th the magnitude of the greater (and therefore equal to that *diff.*), it becomes evident that the arc-length may be expressed in terms of the sine as follows :—Putting A for the *arc*, S for the *sine*, and d for the *difference*, . . .

then $A = S + d = S + \frac{S}{n} + \frac{d}{n} = S + \frac{S}{n} + \frac{S}{n^2} + \frac{d}{n^2} = S + \frac{S}{n} + \frac{S}{n^2} + \frac{S}{n^3} + \frac{d}{n^3}$

$= S + \frac{S}{n} + \frac{S}{n^2} + \frac{S}{n^3} + \frac{S}{n^4} + \frac{d}{n^4} =$ and so on; finally $A = \frac{n}{n-1} S$.

It may be here noted that the relationship now made evident and demonstrated, namely . . that the *diff.* is necessarily an aliquot divisor of the octant's arc-length, at once shows that the ratio of the circle's *circumference* to the *diameter*, at present held by mathematicians to be the true ratio, is *in fact* impossible. For that ratio, taking the radius = 10, gives the octant's arc-length = 7·85398 . . . , and deducting from this 7·07107—for the sine (a quantity which admits of no doubt whatever), we have ·78291 . . . as the quantitative value of the *difference*. Now ·78291 . . . is not an aliquot divisor of 7·85398 . . . . and the actual division by that quantity gives the quotient 10·03178...

It cannot be stated as a *direct* geometrical corollary, but it is *virtually* a corollary and can easily be shown to be a *mathematical* corollary, to the foregoing demonstration that: If the arc-length of an octant be evenly divided into *ten* equal parts the *diff.* is magnitudinally equal to (and *in fact* constitutes) one of those *ten* equal parts.

To make the *certainty* of the conclusion quite evident, the following remarks will here suffice :—

(*Obs.*—The required proof not directly shown by the foregoing demonstration, is that the ratio of one to *n* must necessarily be the ratio of one to *ten*.)

(*a.*) It is already known with perfect certainty that the ratio is very considerably greater than one to *eleven*, and very considerably less than one to *nine;* and that *in fact* it cannot be much greater or less than one to *ten*.

(*b.*) It is easy to mathematically demonstrate the fact stated in (*a*) by methods such as that of comparing the perimeters of inscribed and circumscribed polygons with the circle's circumference.

(*c.*) Careful *mechanical* measurement of the circle's circumference suffices to establish certainty on the point; (viz., that the ratio is greater than one to *eleven*, and less than one to *nine*).

(*d.*) The "Mechanico-Mathematical Instrument" makes evident that the ratio cannot be much greater or much less than one to *ten*.

The complete *direct* answer to the requisition of the Problem :—"It is required to show the lineal ratio of the Octant to its Sine" is as follows :—

(1.) Because the radius of the Octant is a natural *unit* or complete numerical quantity, and is necessarily divisible into *unital increments*, each of them numerically complete in itself, and in quantitative composition similar to the primal unit of which it is an unital increment, therefore the radius of the Octant is naturally divisible into ten equal parts, each of which is naturally divisible into ten equal parts; and so on: that is to say, the radius of the Octant is *decimally* divisible, and can be *only decimally* divided, into unital increments, for the primary unital increment is the one-tenth part, the secondary unital increment is the one-hundredth part, the tertiary unital increment is the one-thousandth part, and so on.

(2.) Because the radius of the Octant is naturally divisible into ten equal parts, the arc and the sine of the Octant are each of them also evenly divisible into ten equal parts.

(3.) Because the *sine* and the *diff.* together compound the arc-length of the octant, and because each unital increment of the arc contains a primary unital increment of the *sine* together with a primary unital increment of the *diff.*, which primary unital increment of the *diff.* is compounded of a secondary unital increment of the *sine* together with a secondary unital increment of the *diff.*, which secondary unital increment of the *diff.* is compounded of a tertiary unital increment of the *sine* together with a tertiary unital increment of the *diff.*, and so on,—therefore, when the arc of the octant is evenly divided into ten parts, the magnitudinal *difference* between the arc and the sine is equal to one of these ten equal parts. *Wherefore :*—The lineal ratio of the octant to its sine is the ratio of *ten* to *nine.*

The values . . of the perimeter of the ultimate inscribed regular polygon, of the circle's circumference, and of the perimeter of the ultimate circumscribed regular polygon, respectively, relatively to the circle's diameter, are, when the diameter = 1, as follows :—

Perimeter of ult^{e} inscribed reg. polygon . .	=	3·141593 . . .
Circumference of the circle	=	3·142696 . . .
Perimeter of ult^{e} circumscribed reg. polygon	=	3·143802 . . .

Notice to the Reader.—The greater part of the foregoing is taken from Mr. Harris's Second Lecture "On the Circle and Straight Line," which will be published very soon, accompanied with numerous diagrams illustrating the structural harmonies in the lineal correlations of the octant's components, and furnishing the general demonstration in an amplified and more complete form.

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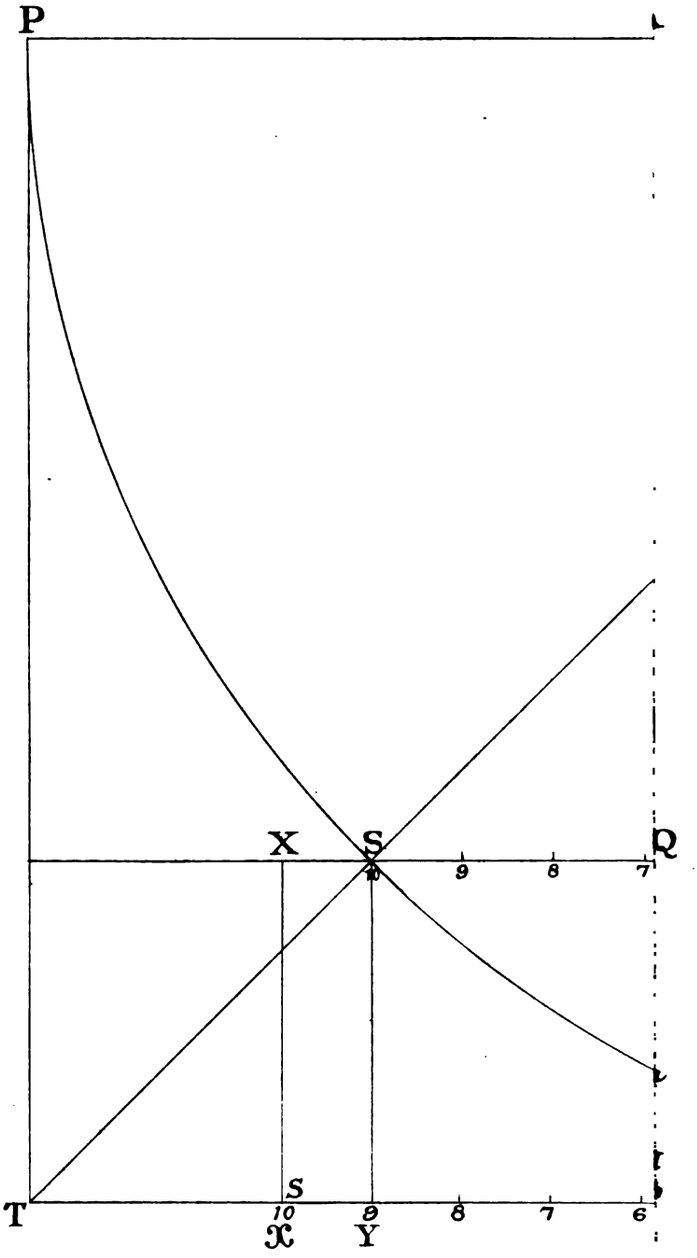


SUPPLEMENTARY DEMONSTRATION that th

Inscribed Square is to the Quadrant of Circle as 9

Postulate—Let it be granted that the ratio is greater than 1 less than 8 to 9.

Fig. 1.—*B F* is a greater quadrant described with radius *A B*. The lesser radius *a b* is the one-*n*th* greater, and the lesser quadrant *b f* described with radius *a b* is consequently one-*n*th of the greater qua The same point *C* (or *c*) bisects both quadrants, a the lesser octants *b c* and *c f*, is equal to one-*n*th of ei the greater octants, *B C* and *C F*. *C Q* and *c q* ar respectively of the greater and lesser octants *B C* and the point *C* (or *c*) draw a perpendicular intercepti *b t* in *d* and the line *B T* in *D*. Let the lesser quadra on the tangential line *b t* until the point *c* becomes on that line, and let *o* indicate the point of contact. represents the lesser octant *c f*. Join *c s*. Now, if a cular *r o* be drawn from *o* through *c s*, evidently *r s*, the octant *o s*, must equal the sine *q c* of the octan point to be determined is the ratio of *c r* (or *d o*), the distance through which the lesser octant advances, to of that octant. Let the greater quadrant be rolled on t tial line *B T*, until the point *C* becomes in contact on Since the quadrant *B F* is *n* times greater than the qu *D O* the horizontal advance of the greater must be *n* ti horizontal advance of the lesser, and the sine-length greater must be *n* times greater than *o y* the sine-len lesser. But *d y* is the arc-length of the octant *o s*, one-*n*th of the octant *O S*. Hence *D O* the *diff.* mu (be wholly compounded of) one-*n*th of itself and

* In the diagram the actual division of the radius is into *ten* p division might be into 9, 8, 11 or 12 parts, and the reasoning wou Hence the postulate.

ore *C R*, the *diff.* of the sine and arc- he arc-length *B O* of the octant *B C*.

When the lesser octant *b c* is rolled on ity *c* arrives at *o*. Then *o R* represents at *c f*: and if the rolling be continued of that second octant be upon the line of arrival, and *y x* must necessarily equal tains the sine-length, and *d y* contains lesser octant. Now *d y* equals *D O* the nt: and the lesser octant, of which *d y* ne-*n*th the magnitude of the greater

e *a b* is one-*n*th of *A B*, *c q* is one-*n*th *D O*, and *B o* (or *B d*) is one-*n*th of *B O*. *à C* contains *n*-1 parts of *C Q* the sine together, and if the arc *à C* be applied *o*, and be then rolled from that point on that line, the point of contact will t *O*, as before; and the advance must con- Hence $(B o \times n\text{-}1) + d o = B D + d o$. of the arc is applied upon *o*, and *B o* in- efore, *n*-1 times *B o* equals *B D*.

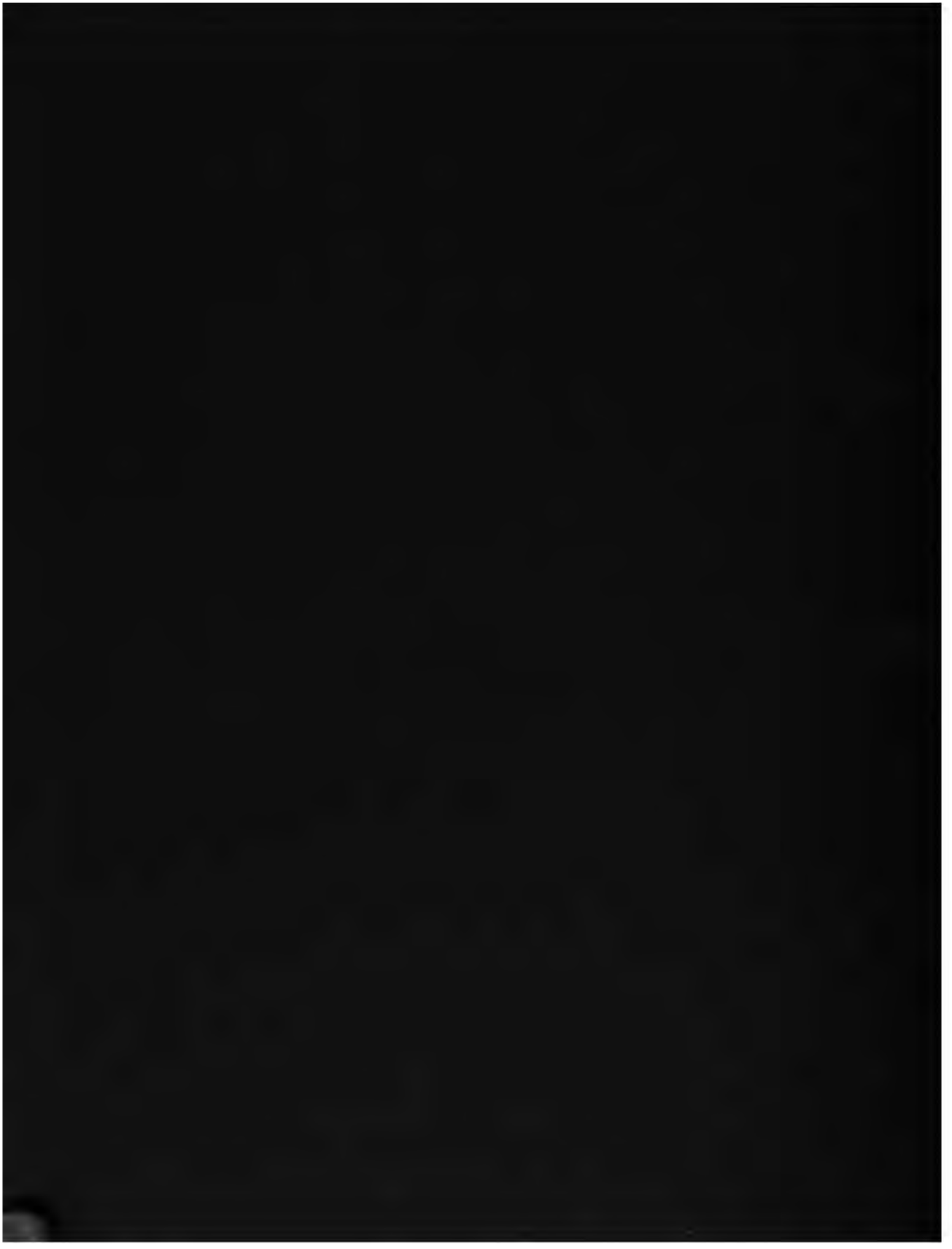
0. And let one-*n*th be the one-tenth. he sine *C Q* (or *O Y*) $= \sqrt{50} = 7{\cdot}071068$. 71068. Wherefore: the arc-length *B C* + ·7071068 + ·07071068 + ·007071068 007071068 + &c., &c., ad infinitum = = 7·85674...

And also
7·071068...
·785674...

7·856742...

8...
68...

&c., the value of the octant's arc-length.



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